T-norm based cuts of IF-sets

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IF-set on X - a pair of mappings $A = (M, N)$
$M, N : X \to [0, 1], M + N \leq 1$
M - membership function
N - nonmembership function

Aims:
- to introduce a suitable concept for α-cuts of IF-sets
- a natural extension of an α-cut B_α for a fuzzy set B

$$B_\alpha = \{x \in X; B(x) \geq \alpha\},$$

in sense that if we consider a fuzzy set B as an IF-set $A = (B, 1 - B)$ then the α-cut A should be equal at least for the most usual case

Original definition

Let $\alpha \in (0, 1]$. The α-cut of the IF-set $A = (M_A, N_A)$ is the set

$$A_\alpha = \{x \in X | M_A(x) \geq \alpha, N_A(x) \leq 1 - \alpha\}$$

Atanassov, K.: Intuitionistic Fuzzy Sets, Theory and Applications, *Physica-Verlag, Heidelberg*, 1999.

$$N_A(x) \leq 1 - \alpha \Longleftrightarrow 1 - N_A(x) \geq \alpha$$

$M_A(x)$ - x belongs to A
$1 - N_A(x)$ - it is not true that x does not belong to A

Interpretation in terms of fuzzy logic

We consider the conjunction of the statements:

- The element x belongs to A,
- It is not true that the element x does not belong to A.

The truth value (in fuzzy logic) of this conjunction should be at least α.

$$T(M_A(x), 1 - N_A(x)) \geq \alpha$$

Definition

If $A = (M_A, N_A)$ is an IF-set on X, if $\alpha \in (0, 1]$ and if T is a left continuous triangular norm, then the T-based α-cut of A is the set

$$A_{T,\alpha} = \{x \in X; T(M_A(x), 1 - N_A(x)) \geq \alpha\}.$$

If $T = T_{min}$ then $A_{T,\alpha}$ coincides with the cuts defined by Atanassov.

Example

Let A be an IF-set on $[0, \infty)$ with the following membership and nonmembership functions:

$$M_A(x) = \begin{cases} 1 - x & \text{for} \quad 0 \leq x \leq 1, \\ 0 & \text{for} \quad x > 1. \end{cases}$$

$$N_A(x) = \begin{cases} \frac{x}{2} & \text{for} \quad 0 \leq x \leq 2, \\ 1 & \text{for} \quad x > 2. \end{cases}$$

If $\alpha = 0.5$, then:

$A_{T_{min},0.5} = [0, \frac{1}{2}]$,

$A_{T_p,0.5} = [0, \frac{3}{2} - \frac{\sqrt{5}}{2}]$,

$A_{T_L,0.5} = [0, \frac{1}{3}]$,

$A_{T_D,0.5} = \{0\}$.

Thus $A_{T_m,0.5} \supseteq A_{T_p,0.5} \supseteq A_{T_L,0.5} \supseteq A_{T_D,0.5}$.

Properties

Monotonicity

Let A be an IF-set on X, let $\alpha, \beta \in (0,1], \alpha > \beta$. Then $A_{T,\alpha} \subseteq A_{T,\beta}$.

$\operatorname{supp} A = \{x \in X; M_A(x) > 0, N_A(x) < 1\}$

Support

Let A be an IF-set on X, let T have no zero divisors. Then $\operatorname{supp} A = \cup\{A_{T,\alpha}; \alpha > 0\}$.

Semicontinuity

Let A be an IF-set on X, let $\alpha \in (0,1]$. Then

$$A_{T,\alpha} = \cap_{\beta<\alpha} A_{T,\beta}.$$

Theorem

Let T be an arbitrary left continuous triangular norm. Let $\{A_\alpha\}_{\alpha\in[0,1]}$ be a system of subsets in X. Then it is a system of T-based cuts of some IF-set in X if and only if it fulfills the following properties:

1. if $\alpha > \beta$ then $A_\alpha \subseteq A_\beta$
2. $A_\alpha = \cap_{\beta<\alpha} A_\beta$ for all $\alpha \in (0, 1]$.

The IF-set constructed from the cuts is not unique, not even for the minimum t-norm - it is sufficient for M_A to take any N_A fulfilling the property $N_A \leq 1 - M_A$. Then the system of cuts remains the same for any such N_A

- different t-norms – various grades of strictness for the conjunction between "to belong" and "not to belong"
- not a surprise that turning a usual fuzzy set A into an IF-set $(A, 1 - A)$ we also get various cuts for different norms
- $A_\alpha = (A, 1 - A)_{T,\alpha}$ holds only for the minimum t-norm

An alternative statement on cuts

- x belongs to A,
- we are able to localise x.

The second statement may be replaced by the disjunction "x belongs to A" or "x does not belong to A" represented by a triangular conorm.

Alternative definition of cuts

Definition

If $A = (M_A, N_A)$ is an IF-set on X, if $\alpha \in (0, 1]$ and if T is a left continuous t-norm with the conorm S_T, then the T-based α-cut of A is the set

$$A_{T,\alpha} = \{x \in X; T(M_A(x), S_T(M_A(x), N_A(x))) \geq \alpha\}.$$

Taking the Łukasiewicz triangular norm we obtain for the IF-set of the type $(M, 1 - M)$ the following:

$$T_L(M(x), S_{T_L}(M(x), 1 - M(x))) = T_L(M(x), 1) = M(x)$$

therefore or an IF-set $(A, 1 - A)$

$$(A, 1 - A)_{T,\alpha} = A_\alpha.$$

Identity between classical cuts and t-norm based cuts

If $T \leq T_L$, then $S_T \geq S_{T_L}$ and so

$$S_{T_L}(M(x), 1 - M(x)) = 1 \Longrightarrow S_T(M(x), 1 - M(x)) = 1.$$

Then for any such t-norm we have

$$T(M(x), S_T(M(x), 1 - M(x))) = T(M(x), 1) = M(x)$$

and so the usual cuts for fuzzy sets, when considered as IF-sets, are saved for any t-norm less or equal to the Łukasiewicz triangular norm.
A direct verification shows that they are saved for T_m, too.
Thus the cuts for fuzzy sets are saved for $[T_D, T_L] \cup \{T_m\}$.

Identity between classical cuts and t-norm based cuts

The complete description of t-norms for which

$$(A, 1-A)_{T,\alpha} = A_\alpha$$

leads to the equation

$$T(\alpha, 1 - T(\alpha, 1-\alpha)) = \alpha.$$

for which the solution is not known in general.

Proposition

Let T be an ordinal sum of t-norms for which an adjusted copy of any t-norm fulfilling $T(\alpha, 1-\alpha) = 0$ for all $\alpha \in [0,1]$ is located in the square $[0.5-c, 0.5+c]^2, c \in (0, 0.5]$. Let the remaining t-norms be T_m on $[0, 0.5-c)^2$ and also on $(0.5+c, 1]^2$. Then

$$T(\alpha, 1 - T(\alpha, 1-\alpha)) = \alpha$$

for all $\alpha \in [0,1]$.

Moreover, as T_m is the only t-norm satisfying $T(\alpha,\alpha) = \alpha$ on the main diagonal, this is the only way how a given t-norm in the square $[0.5-c, 0.5+c]^2, c \in (0, 0.5]$ can be completed to an ordinal sum with required properties.

Solution for ordinal sums