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Similarities and Cardinalities of Interval-valued Fuzzy Sets

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Preliminaries

The lattice $\mathcal{L}^I$

We define $\mathcal{L}^I = (L^I, \leq_{L^I})$ by

$$L^I = \{[x_1, x_2] \mid (x_1, x_2) \in [0,1]^2 \text{ and } x_1 \leq x_2\},$$

$$[x_1, x_2] \leq_{L^I} [y_1, y_2] \iff (x_1 \leq y_1 \text{ and } x_2 \leq y_2),$$

$$\text{for all } [x_1, x_2], [y_1, y_2] \in L^I.$$

Interval-valued fuzzy set

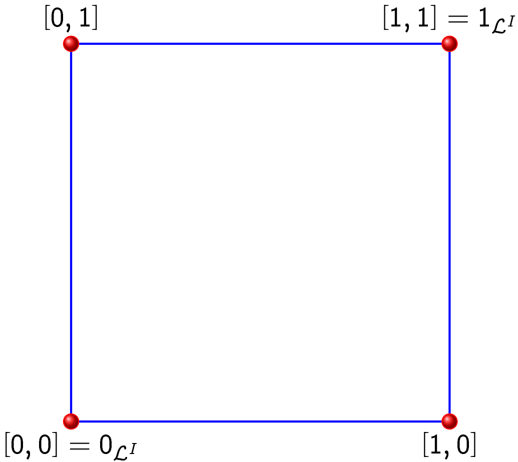
An interval-valued fuzzy set on U is a mapping $A : U \to L^I$.

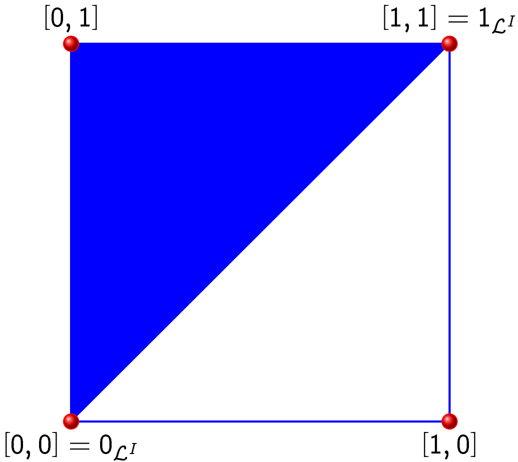
For further usage...

$$\bar{L}^I = \{[x_1, x_2] \mid (x_1, x_2) \in \mathbb{R}^2 \text{ and } x_1 \leq x_2\},$$

$$\bar{L}^I_+ = \{[x_1, x_2] \mid (x_1, x_2) \in [0, +\infty[^2 \text{ and } x_1 \leq x_2\},$$

$$D = \{[x_1, x_1] \mid x_1 \in [0,1]\}.$$


$[0,1]$
$[1,1] = 1_{\mathcal{L}^I}$
$[0,0] = 0_{\mathcal{L}^I}$
$[1,0]$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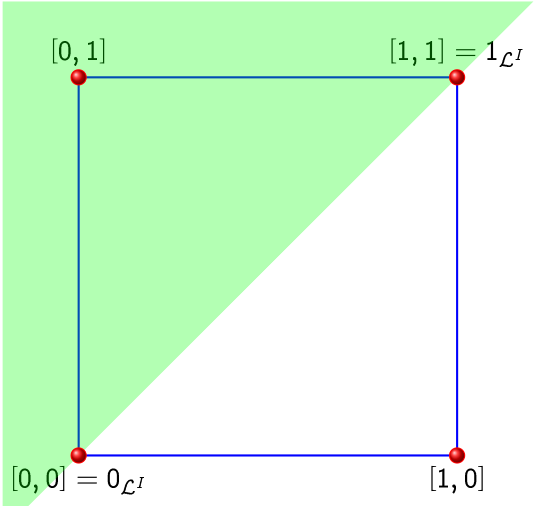
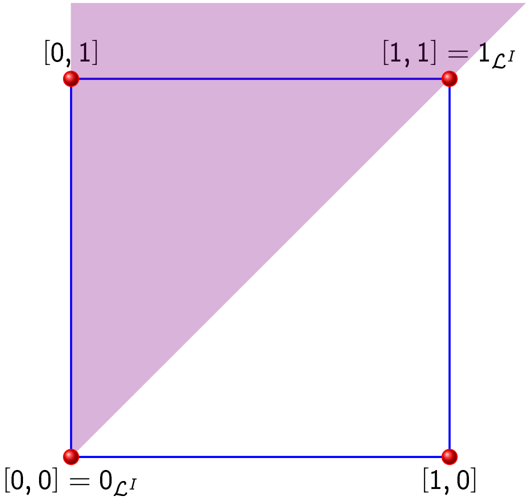
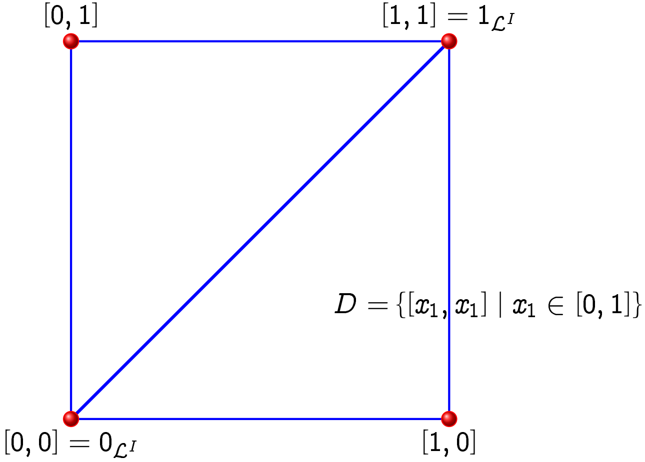


$$\bar{L}^I = \{[x_1, x_2] \mid (x_1, x_2) \in \mathbb{R}^2 \text{ and } x_1 \leq x_2\}$$



$$\bar{L}_+^I = \{[x_1, x_2] \mid (x_1, x_2) \in [0, +\infty[^2 \text{ and } x_1 \leq x_2\}$$



$[0,1]$
$[1,1] = 1_{\mathcal{L}^I}$
$D = \{[x_1, x_1] \mid x_1 \in [0,1]\}$
$[0,0] = 0_{\mathcal{L}^I}$
$[1,0]$

Similarity measure of fuzzy sets

A real function $S : (\mathcal{F}^F(U))^2 \longrightarrow [0, 1]$ is called a similarity measure of fuzzy sets if S satisfies the following properties: for all A, B, C in $\mathcal{F}^F(U)$,

- S1 $S(A, A) = 1$,
- S2 $S(A, B) = S(B, A)$,
- S3 if $A \in \mathcal{P}(U)$, then $S(A, \mathrm{co}\, A) = 0$,
- S4 if $A \subseteq B \subseteq C$, then $S(A, C) \leq \inf(S(A, B), S(B, C))$.

Note that S4 is equivalent to: for all A, B, C and D in $\mathcal{F}(U)$, if $A \subseteq B \subseteq C \subseteq D$, then $S(A, D) \subseteq S(B, C)$.

De Baets et al.

The family of similarity measures:

$$S(A,B) = \frac{a\alpha_{A,B} + b\omega_{A,B} + c\delta_{A,B} + d\nu_{A,B}}{a'\alpha_{A,B} + b'\omega_{A,B} + c'\delta_{A,B} + d'\nu_{A,B}},$$

where $\alpha_{A,B} = \min\{\mathrm{card}(A \setminus B), \mathrm{card}(B \setminus A)\}$, $\omega_{A,B} = \max\{\mathrm{card}(A \setminus B), \mathrm{card}(B \setminus A)\}$, $\delta_{A,B} = \mathrm{card}(A \cap B)$, $\nu_{A,B} = \mathrm{card}(\mathrm{co}(A \cup B))$, with $a, a', b, b', c, c', d, d'$ in $\{0, 1\}$.

T-norms and related operations on $\mathcal{L}^I$

- A t-norm on $\mathcal{L}^I$ is a commutative, associative mapping $\mathcal{T} : (L^I)^2 \to L^I$ which is increasing in both arguments and which satisfies $\mathcal{T}(1_{\mathcal{L}^I}, x) = x$, for all $x \in L^I$.
- A t-conorm on $\mathcal{L}^I$ is a commutative, associative mapping $\mathcal{S} : (L^I)^2 \to L^I$ which is increasing in both arguments and which satisfies $\mathcal{S}(0_{\mathcal{L}^I}, x) = x$, for all $x \in L^I$.
- A negation on $\mathcal{L}^I$ is a decreasing mapping $\mathcal{N} : L^I \to L^I$ which satisfies $\mathcal{N}(0_{\mathcal{L}^I}) = 1_{\mathcal{L}^I}$ and $\mathcal{N}(1_{\mathcal{L}^I}) = 0_{\mathcal{L}^I}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L^I$, then $\mathcal{N}$ is called involutive.

Dual t-conorm

Let $\mathcal{T}$ be a t-norm and $\mathcal{N}$ be an involutive negation on $\mathcal{L}^I$. The mapping $\mathcal{T}^*_{\mathcal{N}} : (L^I)^2 \to L^I$ defined by $\mathcal{T}^*_{\mathcal{N}}(x, y) = \mathcal{N}(\mathcal{T}(\mathcal{N}(x), \mathcal{N}(y)))$, for all x, y in L^I, is a t-conorm on $\mathcal{L}^I$ which is called the dual t-conorm of $\mathcal{T}$ with respect to the negation $\mathcal{N}$.

Intersection, union and complement

The generalized intersection $\cap_{\mathcal{T}}$, union $\cup_{\mathcal{S}}$ and complement $\mathrm{co}_{\mathcal{N}}$ of interval-valued fuzzy sets are defined as follows: for all $A, B \in \mathcal{F}_{\mathcal{L}^I}(U)$ and for all $u \in U$,

$$A \cap_{\mathcal{T}} B(u) = \mathcal{T}(A(u), B(u)),$$
$$A \cup_{\mathcal{S}} B(u) = \mathcal{S}(A(u), B(u)),$$
$$\mathrm{co}_{\mathcal{N}}\, A(u) = \mathcal{N}(A(u)).$$

Arithmetic operators on $\mathcal{L}^I$

Addition and multiplication operator

An addition operator on $\bar{L}^I$ is a mapping $\oplus : (\bar{L}^I)^2 \to \bar{L}^I$ which satisfies the following properties:

1. $\oplus$ is commutative,
2. $\oplus$ is associative,
3. $\oplus$ is increasing in both arguments,
4. $0_{\mathcal{L}^I} \oplus a = a$, for all $a \in \bar{L}^I$,
5. $[\alpha, \alpha] \oplus [\beta, \beta] = [\alpha + \beta, \alpha + \beta]$, for all α, β in $[0, +\infty[$,

A multiplication operator on $\bar{L}^I_+$ is a mapping $\otimes : (\bar{L}^I_+)^2 \to \bar{L}^I$ which satisfies the following properties:

1. $\otimes$ is commutative,
2. $\otimes$ is associative,
3. $\otimes$ is increasing in both arguments,
4. $1_{\mathcal{L}^I} \otimes a = a$, for all $a \in \bar{L}^I_+$,
5. $[\alpha, \alpha] \otimes [\beta, \beta] = [\alpha\beta, \alpha\beta]$, for all α, β in $[0, +\infty[$.

Sometimes we will assume that $\oplus$ and $\otimes$ satisfy the following conditions:

- 5* $[\alpha, \alpha] \oplus b = [\alpha + b_1, \alpha + b_2]$, for all $\alpha \in \mathbb{R}$ and $b \in \bar{L}^I$,
- 5* $[\alpha, \alpha] \otimes b = [\alpha b_1, \alpha b_2]$, for all $\alpha \in [0, +\infty[$ and $b \in \bar{L}^I_+$.

Division operator

Using any multiplication operator $\otimes$, a division operator $\oslash$ can be defined as follows, for all x', y' in $\bar{L}^I_{+,0}$,

$$1_{\mathcal{L}^I} \oslash x' = \left[\frac{1}{x_2'}, \frac{1}{x_1'}\right],$$

and

$$x' \oslash y' = 1_{\mathcal{L}^I} \oslash ((1_{\mathcal{L}^I} \oslash x') \otimes y'),$$

and similarly for the subtraction.

Scalar cardinalities of interval-valued fuzzy sets

Let $\oplus$ be an addition operator on $\bar{L}^I$. A mapping $\mathrm{card}_I : \mathcal{F}^F_{\mathcal{L}^I}(U) \to \bar{L}^I_+$ is called a scalar cardinality of interval-valued fuzzy sets if the following conditions hold:

1. coincidence: for all $u \in U$,

$$\mathrm{card}_I(1_{\mathcal{L}^I}/u) = 1_{\mathcal{L}^I};$$

2. monotonicity: for all $a, b \in L^I$ and $u, v \in U$,

$$a \leq_{L^I} b \implies \mathrm{card}_I(a/u) \leq_{L^I} \mathrm{card}_I(b/v);$$

3. additivity: for all $A, B \in \mathcal{F}^F_{\mathcal{L}^I}(U)$,

$$\mathrm{supp}(A) \cap \mathrm{supp}(B) = \emptyset$$
$$\implies \mathrm{card}_I(A \cup B) = \mathrm{card}_I(A) \oplus \mathrm{card}_I(B).$$

Representation theorem

A mapping $\mathrm{card}_I : \mathcal{F}^F_{\mathcal{L}^I}(U) \to \bar{L}^I_+$ is a scalar cardinality iff there exists a mapping $f_I : L^I \to L^I$ (called scalar cardinality pattern) fulfilling the following conditions:

1. $f_I(0_{\mathcal{L}^I}) = 0_{\mathcal{L}^I}$, $f_I(1_{\mathcal{L}^I}) = 1_{\mathcal{L}^I}$,
2. $f_I(a) \leq_{L^I} f_I(b)$ whenever $a \leq_{L^I} b$,

such that

$$\mathrm{card}_I(A) = \bigoplus_{u \in \mathrm{supp}(A)} f_I(A(u)),$$

for each $A \in \mathcal{F}^F_{\mathcal{L}^I}(U)$.

Similarity measure of interval-valued fuzzy sets

A similarity measure of interval-valued fuzzy sets is a mapping $\mathcal{S} : (\mathcal{F}^F_{\mathcal{L}^I}(U))^2 \longrightarrow [0,1]$ measuring the degree of similarity between interval-valued fuzzy sets if it satisfies the following set of axioms for all A, B, C in $\mathcal{F}^F_{\mathcal{L}^I}(U)$:

1. $\mathcal{S}(A,B) = 1$ if (or alternatively iff) $A = B$,
2. $\mathcal{S}(A,B) = \mathcal{S}(B,A)$,
3. $\mathcal{S}(A,C) \leq_{L^I} \inf(\mathcal{S}(A,B), \mathcal{S}(B,C))$ if $A \subseteq B \subseteq C$.

3a. $S(A,B) \geq_{L^I} S(A,C)$ if $|A(u)_1 - B(u)_1| \leq |A(u)_1 - C(u)_1|$ and $|B(u)_2 - A(u)_2| \leq |C(u)_2 - A(u)_2|$ for all $u \in U$.

For our purposes we introduce an extended modal operator D_α for each $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in [0,1]^n$ as the mapping $D_\alpha : \mathcal{F}^F_{\mathcal{L}^I}(U) \to \mathcal{F}^F_{[0,1]}(U)$ defined by, for all $A \in \mathcal{F}^F_{\mathcal{L}^I}(U)$,

$$D_\alpha A(u_k) = (A(u))_1 + \alpha_k((A(u_k))_2 - (A(u_k))_1), \quad \forall k \in \{1, 2, \ldots, n\}.$$

The requirement (3a) can be rewritten as $\widehat{d}(A(u), B(u)) \leq_{L^I} \widehat{d}(A(u), C(u))$, where $\widehat{d}$ is defined as $\widehat{d}(x, y) = [d_{|.|}(x_1, y_1), d_{|.|}(x_2, y_2)]$, for all x, y in L^I, with $d_{|.|}(x_1, y_1) = |x_1 - y_1|$, for all x_1, y_1 in $[0, 1]$.

A weak similarity measure of the first kind

A similarity measure of interval-valued fuzzy sets is a mapping $\mathcal{S} : \mathcal{F}^F_{\mathcal{L}^I}(U)^2 \to L^I$ if it satisfies the following set of axioms for all A, B, C in $\mathcal{F}^F_{\mathcal{L}^I}(U)$:

1. $\mathcal{S}(A, B) = 1_{\mathcal{L}^I}$ if $A = B$ and A, B are crisp or fuzzy sets,
2. $(\mathcal{S}(A, B))_2 = 1$ if $A = B$ and A, B are neither crisp sets nor fuzzy sets,
3. $\mathcal{S}(A, B) = \mathcal{S}(B, A)$,
4. $\mathcal{S}(A, C) \leq_{L^I} \inf(\mathcal{S}(A, B), \mathcal{S}(B, C))$ if $A \subseteq B \subseteq C$,

A weak similarity measure of the second kind

1. $\mathcal{S}(A, B) = 1_{\mathcal{L}^I}$ if $A = B$ and A, B are crisp or fuzzy sets,
2. $(\mathcal{S}(A, B))_2 = 1$ if $A = B$ and A, B are neither crisp sets nor fuzzy sets,
3. $\mathcal{S}(A, B) = \mathcal{S}(B, A)$,
4. $\mathcal{S}(A, C) \leq_{L^I} \inf(\mathcal{S}(A, B), \mathcal{S}(B, C))$ if $D_{(1,1)}A \subseteq D_{(0,0)}B$ and $D_{(1,1)}B \subseteq D_{(0,0)}C$.

A strong similarity measure

1. $\mathcal{S}(A, B) = 1_{\mathcal{L}^I}$ if $A = B$ and A, B are crisp or fuzzy sets,
2. $(\mathcal{S}(A, B))_2 = 1$ if $A = B$ and A, B are neither crisp sets nor fuzzy sets,
3. $\mathcal{S}(A, B) = \mathcal{S}(B, A)$,
4. $\mathcal{S}(A, B) \geq_{L^I} \mathcal{S}(A, C)$ if $\hat{d}(A(u), B(u)) \leq \hat{d}(A(u), C(u))$ for all $u \in U$.

$$
\begin{aligned}
R_1(A,B) &= \inf(1_{\mathcal{L}^I}, \mathrm{card}_I(A \cap_{\mathcal{T}} B) \\
&\quad \oslash \sup(\mathrm{card}_I(A), \mathrm{card}_I(B))), \\
R_3(A,B) &= \inf(1_{\mathcal{L}^I}, \mathrm{card}_I(A \cap_{\mathcal{T}} B) \\
&\quad \oslash \inf(\mathrm{card}_I(A), \mathrm{card}_I(B))), \\
R_5(A,B) &= \inf(1_{\mathcal{L}^I}, \mathrm{card}_I(A \cap_{\mathcal{T}} B) \\
&\quad \oslash \mathrm{card}_I(A \cup_{\mathcal{S}} B)), \\
R_8(A,B) &= \inf(1_{\mathcal{L}^I}, \sup(\mathrm{card}_I(A), \mathrm{card}_I(B)) \\
&\quad \oslash \mathrm{card}_I(A \cup_{\mathcal{S}} B)), \\
R_{11}(A,B) &= \inf(1_{\mathcal{L}^I}, \inf(\mathrm{card}_I(A), \mathrm{card}_I(B)) \\
&\quad \oslash \sup(\mathrm{card}_I(A), \mathrm{card}_I(B))), \\
R_{14}(A,B) &= \inf(1_{\mathcal{L}^I}, \inf(\mathrm{card}_I(A), \mathrm{card}_I(B)) \\
&\quad \oslash \mathrm{card}_I(A \cup_{\mathcal{S}} B)), \\
R_{16}(A,B) &= 1_{\mathcal{L}^I}.
\end{aligned}
$$

Set difference and symmetric difference

$A \setminus_{\mathcal{T},\mathcal{N}} B = A \cap_{\mathcal{T}} \mathrm{co}_{\mathcal{N}}\, B$ and
$A \Delta_{\mathcal{T},\mathcal{S},\mathcal{N}} B = (A \cap_{\mathcal{T}} \mathrm{co}_{\mathcal{N}}\, B) \cup_{\mathcal{S}} (\mathrm{co}_{\mathcal{N}}\, A \cap_{\mathcal{T}} B)$

$$
\begin{aligned}
R_2(A, B) = {} & \inf(1_{\mathcal{L}^I}, \mathrm{card}_I(\mathrm{co}_{\mathcal{N}}(A \Delta_{\mathcal{T},\mathcal{S},\mathcal{N}} B)) \\
& \oslash \sup(\mathrm{card}_I(\mathrm{co}_{\mathcal{N}}(A \setminus_{\mathcal{T},\mathcal{N}} B)), \\
& \mathrm{card}_I(\mathrm{co}_{\mathcal{N}}(B \setminus_{\mathcal{T},\mathcal{N}} A)))), \\
R_4(A, B) = {} & \inf(1_{\mathcal{L}^I}, \mathrm{card}_I(\mathrm{co}_{\mathcal{N}}(A \Delta_{\mathcal{T},\mathcal{S},\mathcal{N}} B)) \\
& \oslash \inf(\mathrm{card}_I(\mathrm{co}_{\mathcal{N}}(A \setminus_{\mathcal{T},\mathcal{N}} B)), \\
& \mathrm{card}_I(\mathrm{co}_{\mathcal{N}}(B \setminus_{\mathcal{T},\mathcal{N}} A)))), \\
R_6(A, B) = {} & \inf(1_{\mathcal{L}^I}, \mathrm{card}_I\, \mathrm{co}_{\mathcal{N}}(A \Delta_{\mathcal{T},\mathcal{S},\mathcal{N}} B) \oslash n) \\
R_7(A, B) = {} & \inf(1_{\mathcal{L}^I}, \sup(\mathrm{card}_I(A \setminus_{\mathcal{T},\mathcal{N}} B), \\
& \mathrm{card}_I(B \setminus_{\mathcal{T},\mathcal{N}} A)) \oslash \\
& \oslash\, \mathrm{card}_I(A \Delta_{\mathcal{T},\mathcal{S},\mathcal{N}} B))
\end{aligned}
$$

$$
\begin{aligned}
R_9(A,B) &= \inf(1_{\mathcal{L}^I}, \sup(\mathrm{card}_I(\mathrm{co}_{\mathcal{N}}(A \setminus_{\mathcal{T},\mathcal{N}} B)),\\
&\quad \mathrm{card}_I(\mathrm{co}_{\mathcal{N}}(B \setminus_{\mathcal{T},\mathcal{N}} A)))) \oslash n\\
R_{10}(A,B) &= \inf(1_{\mathcal{L}^I}, \inf(\mathrm{card}_I(A \setminus_{\mathcal{T},\mathcal{N}} B),\\
&\quad \mathrm{card}_I(B \setminus_{\mathcal{T},\mathcal{N}} A))\\
&\quad \oslash \sup(\mathrm{card}_I(A \setminus_{\mathcal{T},\mathcal{N}} B), \mathrm{card}_I(B \setminus_{\mathcal{T},\mathcal{N}} A)))\\
R_{12}(A,B) &= \inf(1_{\mathcal{L}^I}, \inf(\mathrm{card}_I(\mathrm{co}_{\mathcal{N}}(A \setminus_{\mathcal{T},\mathcal{N}} B)),\\
&\quad \mathrm{card}_I(\mathrm{co}_{\mathcal{N}}(B \setminus_{\mathcal{T},\mathcal{N}} A)))\\
&\quad \oslash \sup(\mathrm{card}_I(\mathrm{co}_{\mathcal{N}}(A \setminus_{\mathcal{T},\mathcal{N}} B)),\\
&\quad \mathrm{card}_I(\mathrm{co}_{\mathcal{N}}(B \setminus_{\mathcal{T},\mathcal{N}} A))))\\
R_{13}(A,B) &= \inf(1_{\mathcal{L}^I}, \inf(\mathrm{card}_I(A \setminus_{\mathcal{T},\mathcal{N}} B),\\
&\quad \mathrm{card}_I(B \setminus_{\mathcal{T},\mathcal{N}} A))\\
&\quad \oslash \mathrm{card}_I(A \Delta_{\mathcal{T},\mathcal{S},\mathcal{N}} B))\\
R_{15}(A,B) &= \inf(1_{\mathcal{L}^I}, \inf(\mathrm{card}_I(\mathrm{co}_{\mathcal{N}}(A \setminus_{\mathcal{T},\mathcal{N}} B)),\\
&\quad \mathrm{card}_I(\mathrm{co}_{\mathcal{N}}(B \setminus_{\mathcal{T},\mathcal{N}} A))) \oslash n).
\end{aligned}
$$

- $\oplus : (\bar{L}^I)^2 \to \bar{L}^I$ is continuous and satisfies 1–5 and 5*;
- $\otimes : (\bar{L}^I_+)^2 \to \bar{L}^I_+$ is continuous and satisfies 1–5 and 5*;
- $\oslash$ is the division operator derived from $\otimes$ and extended to $\bar{L}^I_+$ in such a way that
  - $x \oslash y \in \bar{D}^{+\infty}_+ \iff (x, y) \in (\bar{D}^{+\infty}_+)^2$, for all x, y in $\bar{L}^I_+$,
  - $x \oslash y = 1_{\mathcal{L}^I} \iff x = y \in \bar{D}$, for all x, y in $\bar{L}^I_+$,
  - $x \oslash y \not\geq_{L^I} 1_{L^I}$, for all $(x, y) \in (\bar{L}^{I,+\infty}_+)^2 \setminus (\bar{D}^{+\infty}_+)^2$ satisfying $x \leq_{L^I} y$, and
  - $0_{\mathcal{L}^I} \oslash 0_{\mathcal{L}^I} = 1_{\mathcal{L}^I}$;
- card_I is a cardinality of interval-valued fuzzy sets;

Theorem

Let $\mathcal{T}$ be a t-norm on $\mathcal{L}^I$. Then the mapping $R_1 : \mathcal{F}^F_{\mathcal{L}^I}(U) \to L^I$, for all A, B in $\mathcal{F}^F_{\mathcal{L}^I}(U)$ given as follows:

$$R_1(A, B) = \inf(1_{\mathcal{L}^I}, \text{card}_I(A \cap_{\mathcal{T}} B) \oslash \sup(\text{card}_I(A), \text{card}_I(B))),$$

is a weak similarity measure of interval-valued fuzzy sets iff $f_I(\mathcal{T}(A(u), A(u))) = f_I(A(u)) \in D$, for all $u \in U$ and $A \in \mathcal{F}^F_{[0,1]}(U)$ with $A \neq \emptyset$.

Corollary 1

Let $\mathcal{T}$ be a t-norm on $\mathcal{L}^I$. Assume that the cardinality pattern f_I associated with card_I satisfies $f_I(a) = f_I(b) \in D \iff a = b \in D$, for all a, b in L^I. Then the mapping $R_1 : \mathcal{F}^F_{\mathcal{L}^I}(U) \to L^I$, for all A, B in $\mathcal{F}^F_{\mathcal{L}^I}(U)$ given as follows:

$$R_1(A, B) = \inf(1_{\mathcal{L}^I}, \text{card}_I(A \cap_{\mathcal{T}} B) \oslash \sup(\text{card}_I(A), \text{card}_I(B))),$$

is a weak similarity measure of interval-valued fuzzy sets iff $\mathcal{T}\big|_D = \inf$.

Corollary 2

Let $\mathcal{T}$ be a t-norm on $\mathcal{L}^I$. Assume that the cardinality pattern f_I associated with card_I is strictly increasing. Then the mapping R_1 is a weak similarity measure of interval-valued fuzzy sets iff $\mathcal{T}\big|_D = \inf$.

Corollary 3

Let $\mathcal{T}$ be a t-norm on $\mathcal{L}^I$, continuous on D and $\mathcal{T} < \inf$ on $D \setminus \{0,1\}$. Then the mapping R_1 is a weak similarity measure of interval-valued fuzzy sets iff $f_I(y) = f_I(x)$ for all x and y in $L_{12}^I = \{x | x \text{ in } L^I \text{ and } x_1 > 0 \text{ and } x_2 < 1\}$.

If we do not drop the property 2

$$(d'(\mathrm{card}_I(A \cap_{\mathcal{T}} A)) \otimes d'(\mathrm{card}_I(A)) \otimes$$
$$\otimes [\sqrt{\frac{(\mathrm{card}_I(A))_1 (\mathrm{card}_I(A))_2}{(\mathrm{card}_I(A \cap_{\mathcal{T}} A))_1 (\mathrm{card}_I(A \cap_{\mathcal{T}} A))_2}}, \sqrt{\frac{(\mathrm{card}_I(A))_1 (\mathrm{card}_I(A))_2}{(\mathrm{card}_I(A \cap_{\mathcal{T}} A))_1 (\mathrm{card}_I(A \cap_{\mathcal{T}} A))_2}}])_1 \leq$$

≤ 1, where the mapping

$d' : \bar{L}^I_{+,0} \to \bar{D}'' = \{[\frac{1}{x_2}, x_2] \mid x_2 \in [1, +\infty[\}$ is defined as follows:

$d'(x) = [\sqrt{\frac{x_1}{x_2}}, \sqrt{\frac{x_2}{x_1}}]$, for all $x \in \bar{L}^I_{+,0}$.