Light graphs theory and related problems

Tomáš Madaras

Institute of Mathematics, PF UPJŠ, Košice

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Overview:

- state of research of local properties of plane graphs before 1997

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- introducing the light graphs

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- introducing the light graphs
- brief history of modern research of light graphs
- some derived and related concepts in study of graph structure

Local properties of plane graphs before 1997

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Every plane triangulation of minimum degree 5 contains a 5-valent vertex adjacent with ≤ 6*-valent vertex.*

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Every plane triangulation of minimum degree 5 contains a 5-valent vertex adjacent with ≤ 6-valent vertex.

Theorem (Franklin 1922)

Every plane triangulation of minimum degree 5 contains a 5-valent vertex adjacent with two ≤ 6-valent vertex.

Theorem (Lebesgue 1940)

Each 3-connected plane graph contains

(a) *a 3-face whose type is one of the following:*

- (i) $(3, i, j), 3 \leq i \leq 6, i \leq j$
- (ii) $(3, 7, i), 7 \leq i \leq 41$
- (iii) $(3, 8, i), 8 \leq i \leq 23$
- (iv) $(3, 9, i), 9 \leq i \leq 17$
- (v) $(3, 10, i), 10 \leq i \leq 14$
- (vi) $(3, 11, i), 11 \leq i \leq 13$
- (vii) $(4, 4, i), 4 \leq i$
- (viii) $(4, 5, i), 5 \leq i \leq 19$
- (ix) $(4, 6, i), 6 \leq i \leq 11$
- (x) $(4, 7, i), 7 \leq i \leq 9$
- (xi) $(5, 5, i), 5 \leq i \leq 9$
- (xii) $(5, 6, i), 6 \leq i \leq 7$

or

(b) *a 4-face whose type is one of the following:*

- (i) $(3, 3, 3, i), 3 \leq i$
- (ii) $(3, 3, 4, i), 4 \leq i \leq 11$
- (iii) $(3, 3, 5, i), 5 \leq i \leq 7$
- (iv) $(3, 4, 3, i), 4 \leq i \leq 11$
- (v) $(3, 4, 4, i), 4 \leq i \leq 5$
- (vi) $(3, 4, 5, 4)$
- (vii) $(3, 5, 3, i), 5 \leq i \leq 7$

or

(c) *a 5-face of type* $(3, 3, 3, 3, i), 3 \leq i \leq 5$.

Corollary

Each 3-connected plane graph contains an edge incident with a face of size at most 5 such that sum of degrees of endvertices of this edge is at most 14.

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Theorem (Kotzig 1955)

Every 3-connected plane graph contains an edge such that sum of degrees of its endvertices is at most 13, and at most 11 in the case of absence of 3-valent vertices. The bounds 13 and 11 are best possible.

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Theorem (Borodin 1989)

Every plane graph of minimum degree 5 contains a triangular face such that sum of degrees of its vertices is at most 17. The bound 17 is best possible.

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Introducing light graphs

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Theorem (Fabrici and Jendrol' 1997)

Each 3-connected plane graph G that contains a k-vertex path, contains also a k-vertex path such that each its vertex is of degree at most $5k$ in G. The bound $5k$ is best possible.

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What is the common feature of these results ?

All these results obey the following common form:

Statement:

Every graph G from some family $\mathcal{H}$ of plane graphs contains certain subgraph H such that sum of degrees of this subgraph is "small".

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Every graph G from some family $\mathcal{H}$ of plane graphs contains certain subgraph H such that sum of degrees of this subgraph is "small".

Here "small" means being bounded by some constant that is the same for **all graphs** $G \in \mathcal{H}$.

Definition

Let $\mathcal{H}$ be a family of graphs and let H be a connected graph such that at least one member of $\mathcal{H}$ contains a subgraph isomorphic to H. Let $\varphi(H, \mathcal{H})$ be the smallest integer with the property that each graph $G \in \mathcal{H}$ which contains a subgraph isomorphic to H, contains also a subgraph $K \cong H$ such that

$$(\forall x \in V(K))\ \deg_G(x) \leq \varphi(H, \mathcal{H}).$$

If such an integer does not exist, we put $\varphi(H, \mathcal{H}) = +\infty$.

Definition

Similarly, let $w(H, \mathcal{H})$ be the smallest integer such that each graph $G \in \mathcal{H}$ containing a subgraph isomorphic to H, contains also a subgraph $K \cong H$ such that

$$\sum_{x \in V(K)} \deg_G(x) \leq w(H, \mathcal{H}).$$

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If such an integer does not exist, we put $w(H,\mathcal{H}) = +\infty$.

We say that the graph H is *light* in the family $\mathcal{H}$ if $\varphi(H,\mathcal{H}) < +\infty$ (or, equivalently, $w(H,\mathcal{H}) < +\infty$).

Notation:

P_k	...	k-vertex path
C_k	...	k-vertex cycle
S_k	...	$K_{1,k}$
$\mathcal{P}$	...	family of all plane graphs
$\mathcal{P}_c(\delta, \rho)$	...	family of all c-connected plane graphs of minimum degree $\geq \delta$ and minimum face size $\geq \rho$
$\mathcal{T}(\delta)$	...	family of all plane triangulations of minimum degree $\geq \delta$

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- Kotzig: $w(K_2, \mathcal{P}(3,3)) = 13, w(K_2, \mathcal{P}(4,3)) = 11$

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- Borodin: $w(K_3, \mathcal{P}_1(5,3)) = 17$

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- Kotzig: $w(K_2, \mathcal{P}(3,3)) = 13, w(K_2, \mathcal{P}(4,3)) = 11$
- Borodin: $w(K_3, \mathcal{P}_1(5,3)) = 17$
- Fabrici and Jendrol': $\varphi(P_k, \mathcal{P}_3(3,3)) = 5k$

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Theorem (Fabrici and Jendrol' 1997)

For each integer m and each plane graph H which is not a path, there exists a 3-connected plane graph G_m such that each its subgraph $K \cong H$ contains a vertex of degree at least m in G_m.

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For each integer m and each plane graph H which is not a path, there exists a 3-connected plane graph G_m such that each its subgraph $K \cong H$ contains a vertex of degree at least m in G_m.

Hence, for the family of 3-connected plane graphs, the set of light graphs is "trivial".

Families with complete characterization of light graphs:

Family	Light graphs	Value of φ	Heavy graphs	References
$\mathcal{P}_3(3,3)$	P_k	$5k$	all other	Fabrici, Jendroľ 1997
$\mathcal{P}_3(4,3)$	P_k	$5k-7$ for $k \geq 8$ $4k-1$ for $4 \leq k \leq 7$ $2k+3$ for $2 \leq k \leq 3$	all other	Fabrici, Hexel, Jendroľ, Walther 1999
$\mathcal{P}_3(3,4)$	P_k	$\leq \frac{5}{2}k$	all other	Harant, Jendroľ, Tkáč 1999
$\mathcal{P}_4(4,3)$	P_k	$\leq 2k+3$	all other	Hexel, Walther 1999 Mohar 2000
$\mathcal{P}_2(3,3)$	K_1 K_2	5 10	all other	Kotzig 1955 Jendrol' 1997
$\mathcal{P}_2(4,3)$	K_1 K_2 P_3 P_4	4 7 9 ≤ 191	all other	Kotzig 1955 Jendrol' 1999 T.M., Škrekovski 2004

The family $\mathcal{P}_1(5,3)$:

Light graphs	Value of φ	Value of w	Heavy graphs	References
K_1	5	5		
K_2	6	11		Wernicke 1904
P_3	6	17		Franklin 1922
P_4	7	23		Jendrol' 1999
				Jendrol', T.M. 1996
P_5	≤ 9	29		Jendrol' 1999;
				Mičová, T.M. 2003
S_3	7	23		Jendrol', T.M. 1996
S_4	10	30		Jendrol', T.M. 1996;
				Borodin, Woodall 1998
C_3	7	17		Borodin 1989
C_4	11			Soták
C_5	10			
C_6	≤ 107			Mohar, Škrekovski, Voss 2004
C_7	≤ 359			T.M., Škrekovski, Voss 2007
some other small graphs				T.M., Soták
			all with $\Delta(H) \geq 5$	Fabrici 2002

Observe the discrepancy between the family $\mathcal{P}_3(4,3)$ and $\mathcal{P}_3(5,3)$ - the first yields only "trivial" set of light graphs (just paths), while the second a wide variety of light graphs other than paths.

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Mohar, Škrekovski and Voss (2004) suggested to explore the space "in between", that is, the family of plane graphs of minimum degree at least 4 and minimum edge weight at least 9 (or, informally, with the "minimum degree" 4.5):

Light graphs	Value of φ	Value of w	Heavy graphs
...	...	...	...
P_3		17	
P_4		23	
C_3	21		
C_4	≤ 22	≤ 35	
C_5	≤ 107		
C_6	≤ 107		
S_3		23	
S_4	≤ 107		
			P_k for $k \geq 8$
			S_k for $k \geq 5$
			C_k for $k \geq 7$

When relaxing the condition on minimum vertex degree and considering just the minimum edge weight constraint, it is also possible to obtain results with nontrivial light graphs:

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Theorem (T.M., Škrekovski 2004)

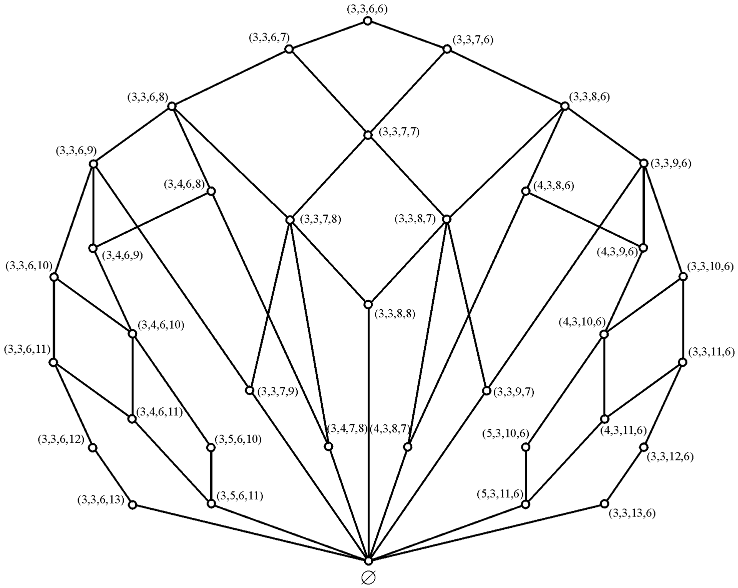
Let $\mathcal{R}(w)$ be the family of all plane graphs of minimum degree at least 3 and minimum edge weight at least w.

1. S_4 *is light in* $\mathcal{R}(w)$ *if and only if* $9 \leq w \leq 13$
2. C_3 *(*C_4*) is light in* $\mathcal{R}(w)$ *if and only if* $10 \leq w \leq 13$
3. P_4 *is light in* $\mathcal{R}(w)$ *if and only if* $8 \leq w \leq 13$.

The similar situation and discrepancy appears when considering the family $\mathcal{P}_3(3,5)$:

Light graphs	Value of φ	Value of w	Heavy graphs
...	...	...	...
C_5	5	17	Lebesgue 1940
		C_k for $k > 5, k \neq 14$	Jendrol', Owens 2001
S_3		13	Madaras 2004
several other small graphs			Madaras 2007
C_5+ path P_k	$90k$		Hajduk, Soták 2006

In general, a nontrivial set of light graphs may be enforced by mutual combination of four constraints: minimum vertex degree $\geq \delta$, minimum face size $\geq \rho$, minimum edge weight $\geq w$ and minimum dual edge weight $\geq w^*$. There are exactly 35 quadruples (δ, ρ, w, w^*) for which the corresponding family $\mathcal{P}(\delta, \rho, w, w^*)$ is nonempty.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(3,3,6,13)
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∅

Selected results on light graphs

Family	Light graphs	Value of φ	Heavy graphs	References
$\mathcal{P}(3,3,6,12)$	C_3	4		Ferencová, T.M. 2007
$\mathcal{P}(3,3,6,13)$	C_{10}	≤ 5	C_r for $4 \leq r \leq 9$	
$\mathcal{P}(3,5,6,11)$	C_6			T.M. 2004
			C_7	Ferencová, T.M. 2007
			C_8	
	C_9			T.M. 2004
	C_{10}			Ferencová, T.M. 2007
$\mathcal{P}(3,3,7,9)$	C_3	≤ 6		Ferencová, T.M. 2007
$\mathcal{P}(3,4,7,8)$	C_4	≤ 11		
$\mathcal{P}(3,3,8,8)$			C_3	
			C_4	
			C_5	
		C_6		

There are other conditions which may enforce nontrivial light graphs:

- minimum degree and minimum weight of prescribed subgraph (other than edge): an example - plane triangulations of minimum degree 5 and minimum triangle weight 17 (T.M., Fabrici, Zlámalová 2007)

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- minimum degree and minimum weight of prescribed subgraph (other than edge): an example - plane triangulations of minimum degree 5 and minimum triangle weight 17 (T.M., Fabrici, Zlámalová 2007)
- excluding cycles of specified length (Fijavž, T.M. - unpublished)

Light graphs in families of nonplanar graphs

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On the other hand, a variety of light structures may be also found in graphs drawn in the plane with crossings.

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Each 1-planar graph contains a vertex of degree at most 7; the bound 7 is best possible.

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Lemma (Ringel 1965)

Each 1-planar graph contains a vertex of degree at most 7; the bound 7 is best possible.

Theorem (Fabrici, T.M. 2007)

Each 3-connected 1-planar graph contains an edge such that its endvertices are of degree at most 20. The bound 20 is best possible.

Light graphs in families of nonplanar graphs

Family	Light graphs	Value of φ	Heavy graphs	References
$\overline{\mathcal{P}}_5$			on > 4 vertices $K_4, K_4^-, K_{1,3}^+$ C_3, C_4	Fabrici, T.M. 2007
$\overline{\mathcal{P}}_5$ with girth 4	C_4 $K_{1,4}$	≤ 9 ≤ 11		D. Hudák, T.M. 2008
$\overline{\mathcal{P}}_6$	C_3 $K_{1,3}$ $K_{1,4}$	10 ≤ 15 ≤ 23	on > 6 vertices $K_6 - 2K_2$	Fabrici, T.M. 2007
$\overline{\mathcal{P}}_7$	$K_{1,5}$ $K_{1,6}$ K_4 $K_{2,3}^{\star}$	≤ 11 ≤ 15 ≤ 13 ≤ 13		Fabrici, T.M. 2007 D. Hudák, T.M. 2008

Light sets of graphs

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Instead of one fixed graph, one may specify a finite set of graphs and look for isomorphic copies of some graphs from this set:

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Instead of one fixed graph, one may specify a finite set of graphs and look for isomorphic copies of some graphs from this set:

Theorem (Appel, Haken)

Each plane triangulation of minimum degree 5 contains either two adjacent 5-vertices or a triangular face of weight 17.

Again, the definition of light set of graphs was inspired by the following general results:

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Theorem (Fabrici and Jendrol' 1998)

Each 3-connected plane graph G on at least $k \geq 3$ vertices contains a connected k-vertex subgraph K such that each its vertex is of degree at most $4k+3$ in G. The bound $4k+3$ is best possible.

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Theorem (Enomoto and Ota 1999)

Each 3-connected plane graph G on at least $k \geq 3$ vertices contains a connected k-vertex subgraph K of weight at most $8k-1$.

Definition

Let $\mathcal{G}$ be a family of graphs and let $\mathcal{H}$ be a finite set of graphs with the property that each graph of $\mathcal{G}$ contains a proper subgraph isomorphic to at least one member of $\mathcal{H}$. Let $\tau(\mathcal{H}, \mathcal{G})$ be the smallest integer with the property that every graph $G \in \mathcal{G}$ contains a subgraph K which is isomorphic to one of the elements in $\mathcal{H}$ such that, for every vertex $v \in V(K)$,

$$\deg_G(v) \leq \tau(\mathcal{H}, \mathcal{G}).$$

If such a finite $\tau(\mathcal{H}, \mathcal{G})$ does not exist we write $\tau(\mathcal{H}, \mathcal{G}) = +\infty$.

Definition

Similarly, let $f(\mathcal{H},\mathcal{G})$ be the smallest integer with the property that every graph $G \in \mathcal{G}$ contains a subgraph K which is isomorphic to one of the elements in $\mathcal{H}$ such that

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If such a finite number does not exist we write $f(\mathcal{H},\mathcal{G}) = +\infty$.

The set $\mathcal{H}$ is *light in the family* $\mathcal{G}$ if $\tau(\mathcal{H},\mathcal{G}) < +\infty$ (or $f(\mathcal{H},\mathcal{G}) < +\infty$).

If we denote the set of all k-vertex trees as $\mathcal{T}_k$, then the results above translate, using defined formalism, as

$$\tau(\mathcal{T}_k, \mathcal{P}_3(3,3)) = 4k+3, f(\mathcal{T}_k, \mathcal{P}_3(3,3)) \leq 8k-1$$

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Theorem (Jendrol' and Voss 2004)

For $S = \{P_k, K_{1,3}\}$, $\tau(S, \mathcal{P}_3(3,3)) = 4k+3$.

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Let S be a finite family of connected plane graphs H such that $\Delta(H) \geq 3$ or $\delta(H) \geq 2$. Then S is not light in $\mathcal{P}_3(3,3)$.

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Let S *be a finite family of connected plane graphs* H *such that* $\Delta(H) \geq 3$ *or* $\delta(H) \geq 2$. *Then* S *is not light in* $\mathcal{P}_3(3,3)$.

Theorem (Fabrici 2002)

$\tau(\mathcal{T}_k, \mathcal{P}_3(4,3)) = 4k-1$ *for* $k \geq 4$.

We also studied light sets comprised of cycles:

Theorem (T.M. 2004)

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Theorem (T.M. 2007)

$\tau(\{C_9, C_{11}\}, \mathcal{P}_3(3,5)) \leq 23$.

Note that neither one of C_8, C_9, C_{11} is light in $\mathcal{P}_3(3,5)$.

Light induced graphs

Considering variation of the light graph definition, one may look for **induced** copies of given graph that are light:

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Theorem (T.M. 2007)

Each 3-connected plane graph contains an induced 3-path whose sum of degrees of vertices is at most 17. The bound 17 is best possible.

Definition

Let $\mathcal{H}$ be a family of graphs and let H be a connected graph such that at least one member of $\mathcal{H}$ contains an induced subgraph isomorphic to H. Let $\varphi_I(H,\mathcal{H})$ be the smallest integer with the property that each graph $G \in \mathcal{H}$ which contains an induced subgraph isomorphic to H, contains also an induced subgraph $K \cong H$ such that

$$(\forall x \in V(K))\ \deg_G(x) \leq \varphi_I(H,\mathcal{H}).$$

If such an integer does not exist, we put $\varphi_I(H,\mathcal{H}) = +\infty$.

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Similarly, let $w_I(H,\mathcal{H})$ be the smallest integer with the property that each graph $G \in \mathcal{H}$ which contains an induced subgraph isomorphic to H, contains also an induced subgraph $K \cong H$ such that

$$\sum_{x \in V(K)} \deg_G(x) \leq w_I(H,\mathcal{H}).$$

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Theorem (R. Soták, T.M.)

A graph H is induced-light in the family $\mathcal{P}(3,3)$ if and only if $H \cong P_k$.

Gravity of a graph in a family

If a graph H is heavy in a family $\mathcal{H}$ then, for every integer m, there exists a graph $G_m \in \mathcal{H}$ such that each isomorphic copy of H in G_m contains **at least one** vertex of degree at least m in G_m.

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For given heavy graph H in $\mathcal{H}$, an integer m and an integer $k \in [1, |V(H)|]$, does there exist a graph $G_m \in \mathcal{H}$ such that each isomorphic copy of H in G_m contains **at least** k heavy vertices ?

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Definition (Madaras and Škrekovski 2007)

The *gravity* $g(H, \mathcal{H})$ of a connected graph H in the family $\mathcal{H}$ is the largest integer k such that, for every integer m, there exists a graph$G_m \in \mathcal{H}, G_m \supseteq H$ such that each isomorphic copy of H in G_m contains at least k vertices of degree at least m in G_m.

Theorem (Madaras and Škrekovski 2007)

$$g(P_n, \mathcal{P}) = \begin{cases} n-3, & n \in \{3,5\} \\ n-2 & \textit{otherwise.} \end{cases}$$

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Theorem (Madaras and Škrekovski 2007)

$g(P_n, \mathcal{P}_2^*) = n - o(n)$ *for infinitely many* n.

Considering a hierarchy of graphs according to their gravity in given family, at the bottom level, there are light graphs. At the next level, there are heavy graphs whose all connected subgraphs are light. Such graphs are called *almost light*.

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Theorem (Madaras and Škrekovski 2007)

The only almost light graph in $\mathcal{P}_3$ is K_2.
In the family of graphs of $\mathcal{P}_3$ having minimum edge weight at least 7, there are two almost light graphs, P_4 and $K_{1,3}$.
In $\mathcal{P}_4$, there are three almost light graphs: C_3, $K_{1,3}$ and P_5.

At the top of the previously mentioned hierarchy, there are graphs H such that $g(H, \mathcal{H}) = |V(H)|$ (*absolutely heavy graphs*).

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Which cycles are absolutely heavy in $\mathcal{P}^*$?

Light graphs incident with small faces

Plane graphs contain not only small degree vertices, but also small degree vertices incident with small faces: in 1940, H. Lebesgue proved that each 3-connected plane graph contains a vertex of degree at most 5 which is incident with a face of size at most 5.

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In general, one cannot guarantee the existence of small vertices incident **only with small faces**, as seen from the pyramide and the antiprism graph.

However, the result of Lebesgue on face types imply that each 3-connected plane graph of minimum degree 5 contains a 5-vertex incident with four triangular faces and one face of size at most 5 (the face size 5 is best possible).

Theorem (T.M. 2004)

Each 3-connected plane graph of minimum face size 5 contains a 5-face adjacent to 5- or 6-face such that all their vertices are of degree at most 9.

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In dual form, this means that each 3-connected plane graph of minimum degree 5 contains a light edge and a light 3-path (as in theorems of Wernicke and Franklin) which are incident only with faces of size at most 9 and 23, respectively.

Definition

Let $\mathcal{H}$ be a family of plane (or, generally, embedded) graphs and let H be a connected graph being a subgraph of at least one member of H. Let $\Phi(H, \mathcal{H})$ be the lexicographic minimum of all pairs (a, b) of integers such that each graph $G \in \mathcal{H}$ containing H contains also a subgraph $K \cong H$ such that $\deg_G(x) \leq a$ and $\deg_G(\alpha) \leq b$ for each $x \in V(K)$ and each face $\alpha \in F(G)$ incident with x. If one of a, b does not exist, we put the corresponding component of $\Phi(H, \mathcal{H})$ equal to $+\infty$.

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The graph H is *doubly light* in the family $\mathcal{H}$ if both components of $\Phi(H, \mathcal{H})$ are finite.

Alternatively, we may consider the requirement of bounded size only for those faces of G that are incident with an *edge* of K. This yield a notion of *weakly doubly light* graph.

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- paths (of certain length) are not even weakly doubly light in $\mathcal{P}_3(\delta, \rho)$
- C_3 is not doubly light in $\mathcal{P}_3(5, 3)$, but it is weakly doubly light in this family
- C_3 is doubly light in $\mathcal{P}(5, 3, 11, 6)$ with $\Phi(C_3, \mathcal{P}(5, 3, 11, 6)) \leq (7, 5)$

Thanks for your attention :-)