Dynamics of a Map of a Triangle and Some Open Problems in Theory of Numbers

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Problem to be solved

Given the plane triangle

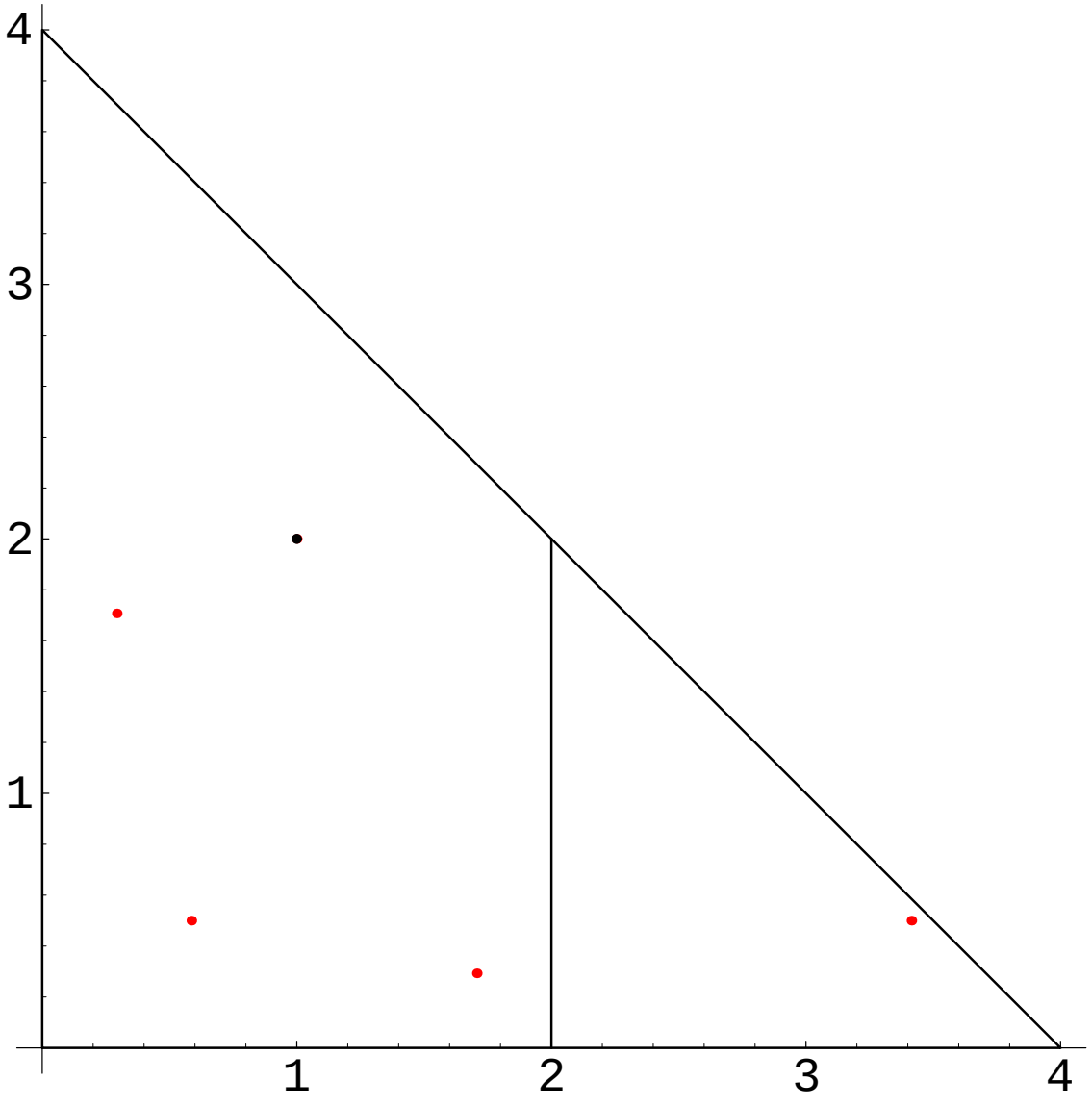
$$D = \{\, [x, y] : 0 \leq x,\ 0 \leq y,\ x + y \leq 4 \,\}$$

we consider the map

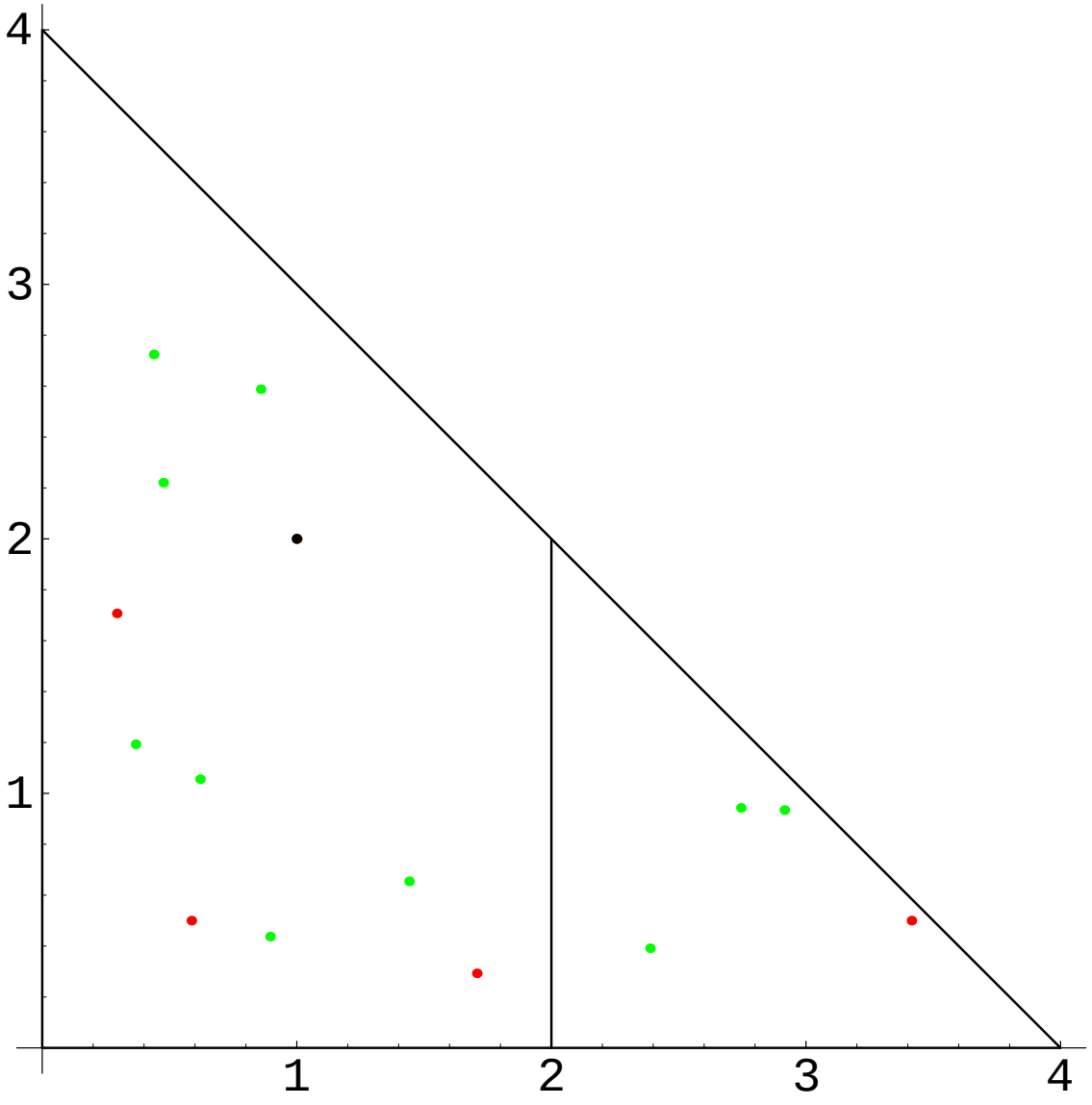
$$F : D \to D,\ [x, y] \mapsto [x(4 - x - y),\ xy]$$

and its periodic points.

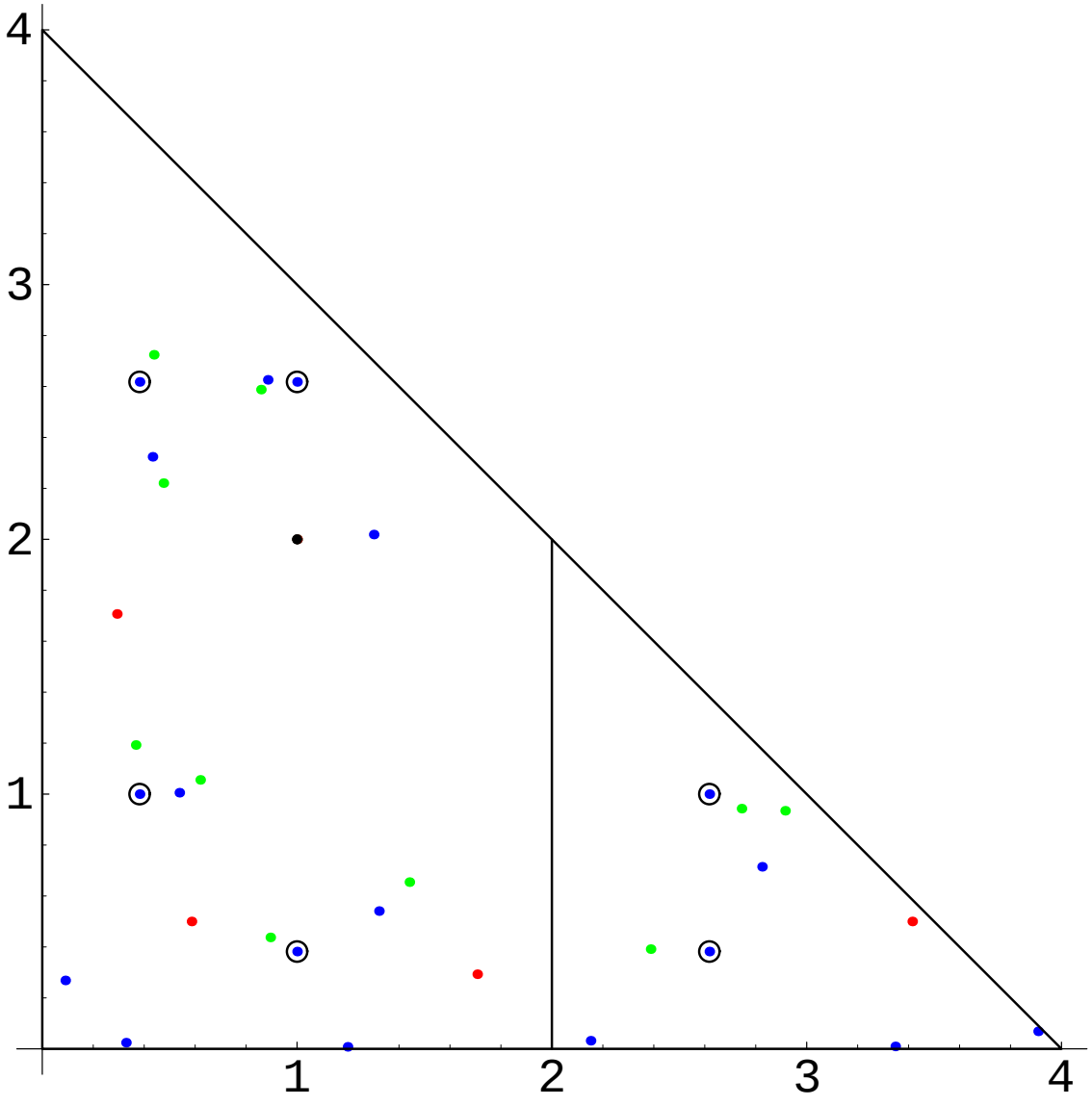
Interior points with period $n \leq 4$



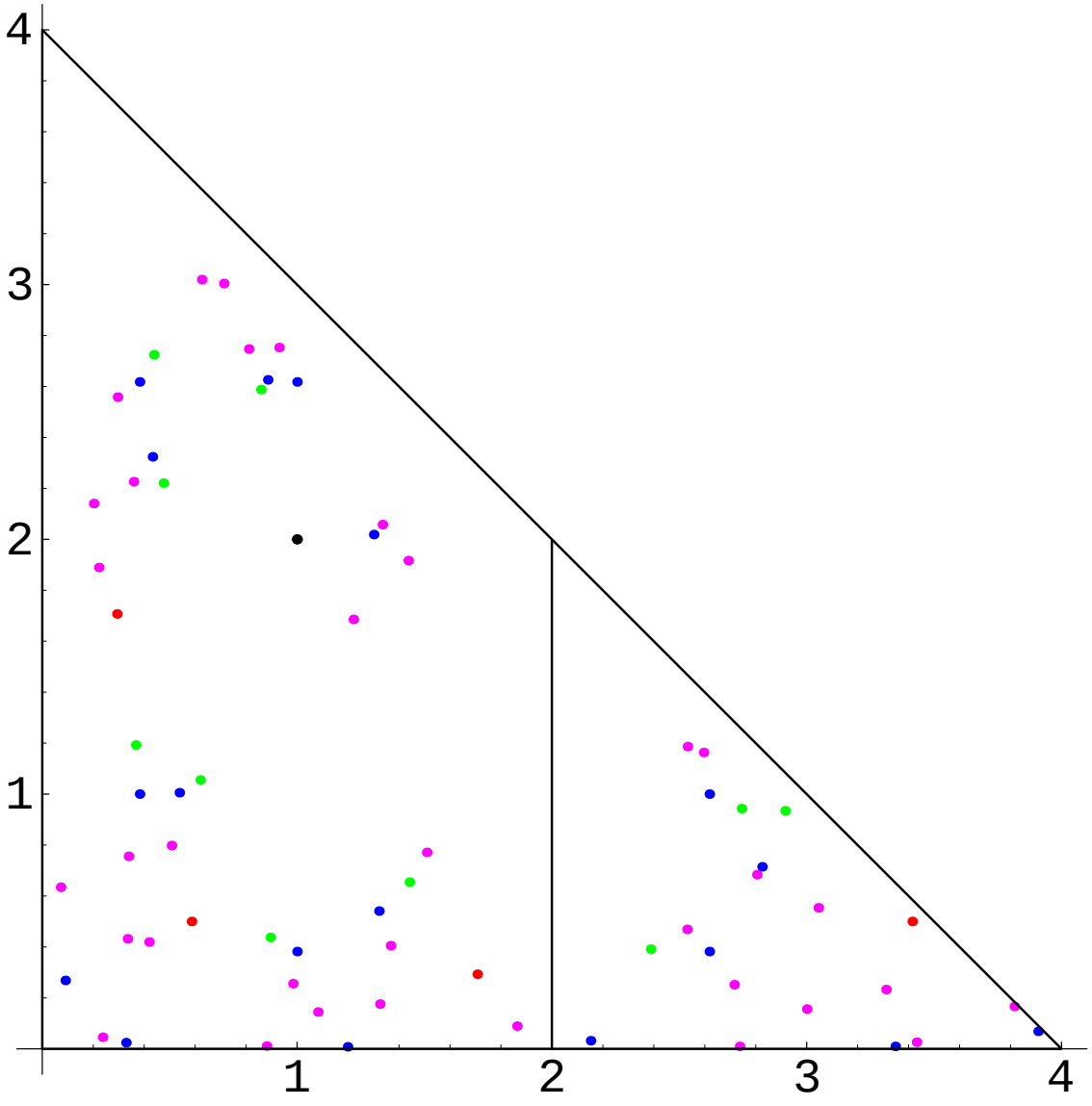
Interior points with period $n \leq 5$



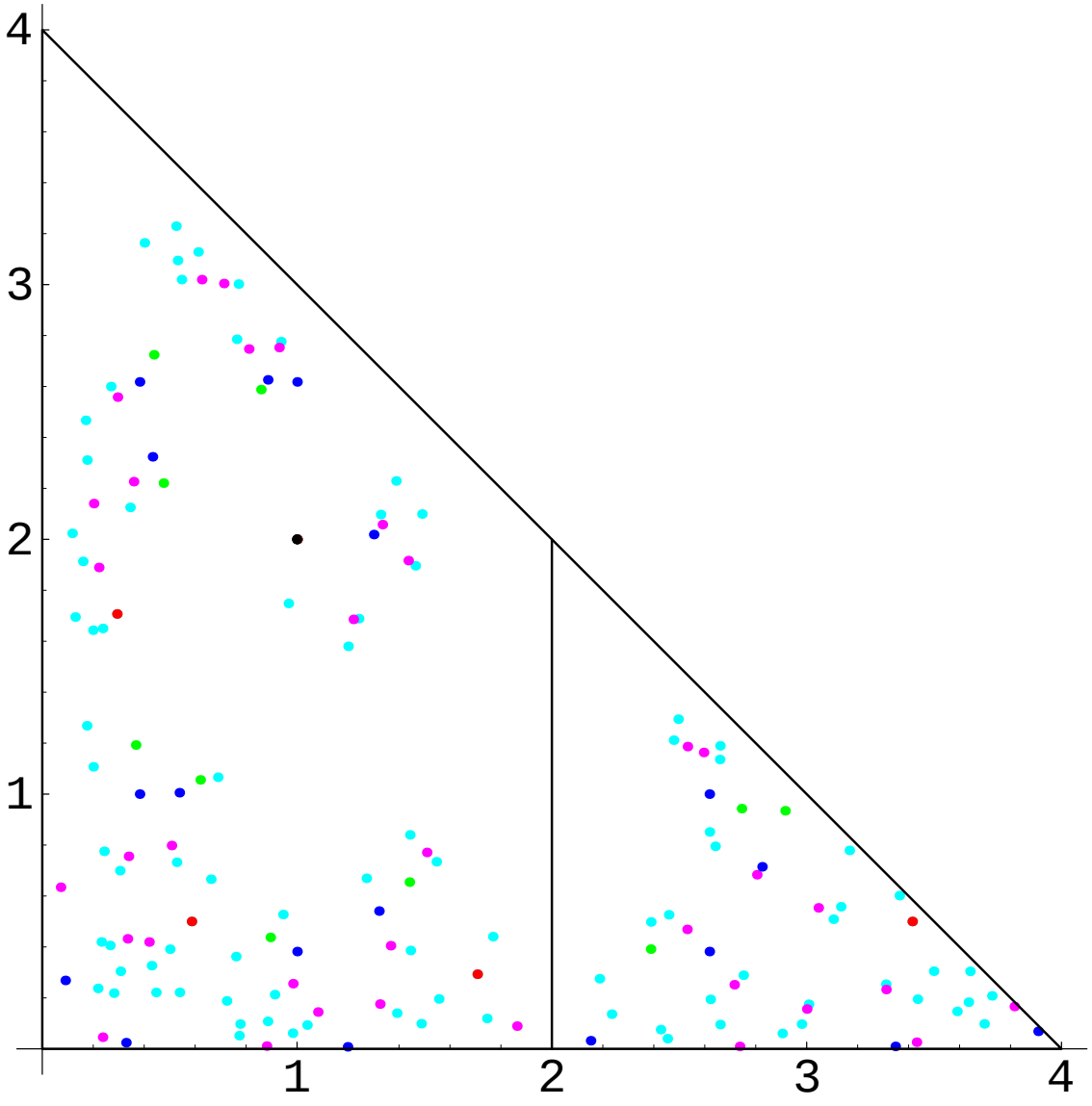
Interior points with period $n \leq 6$



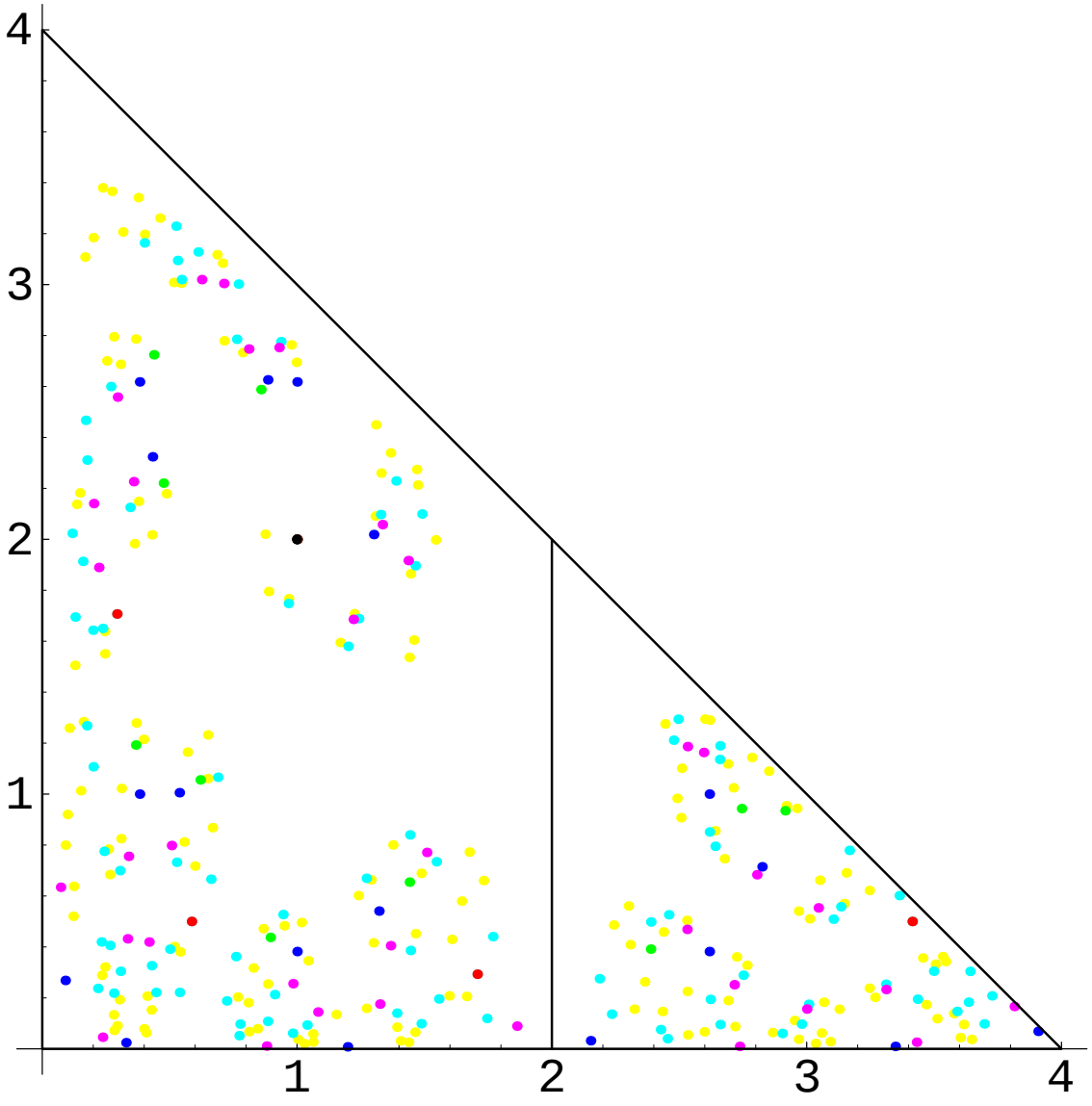
Interior points with period $n \leq 7$



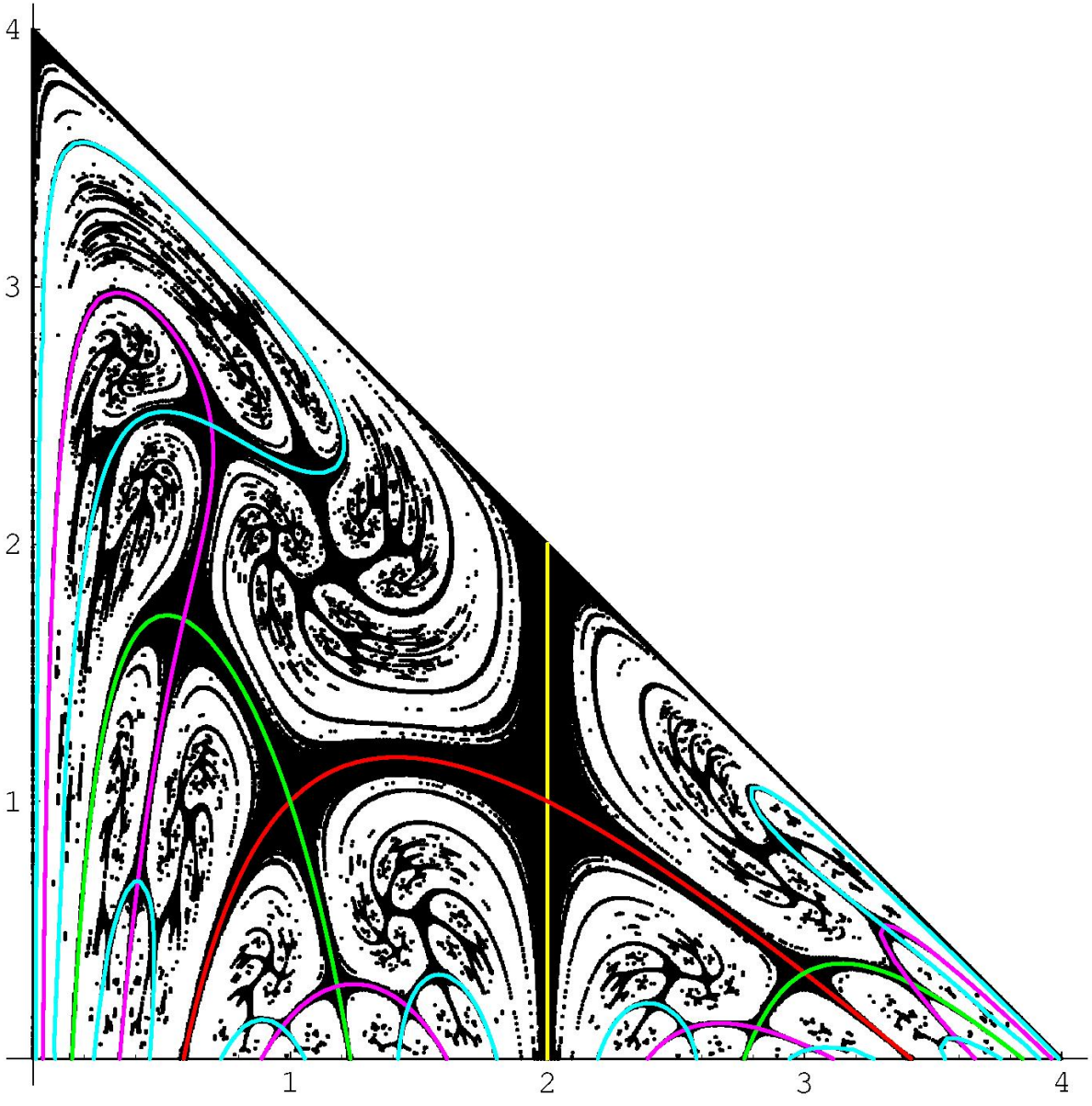
Interior points with period $n \leq 8$



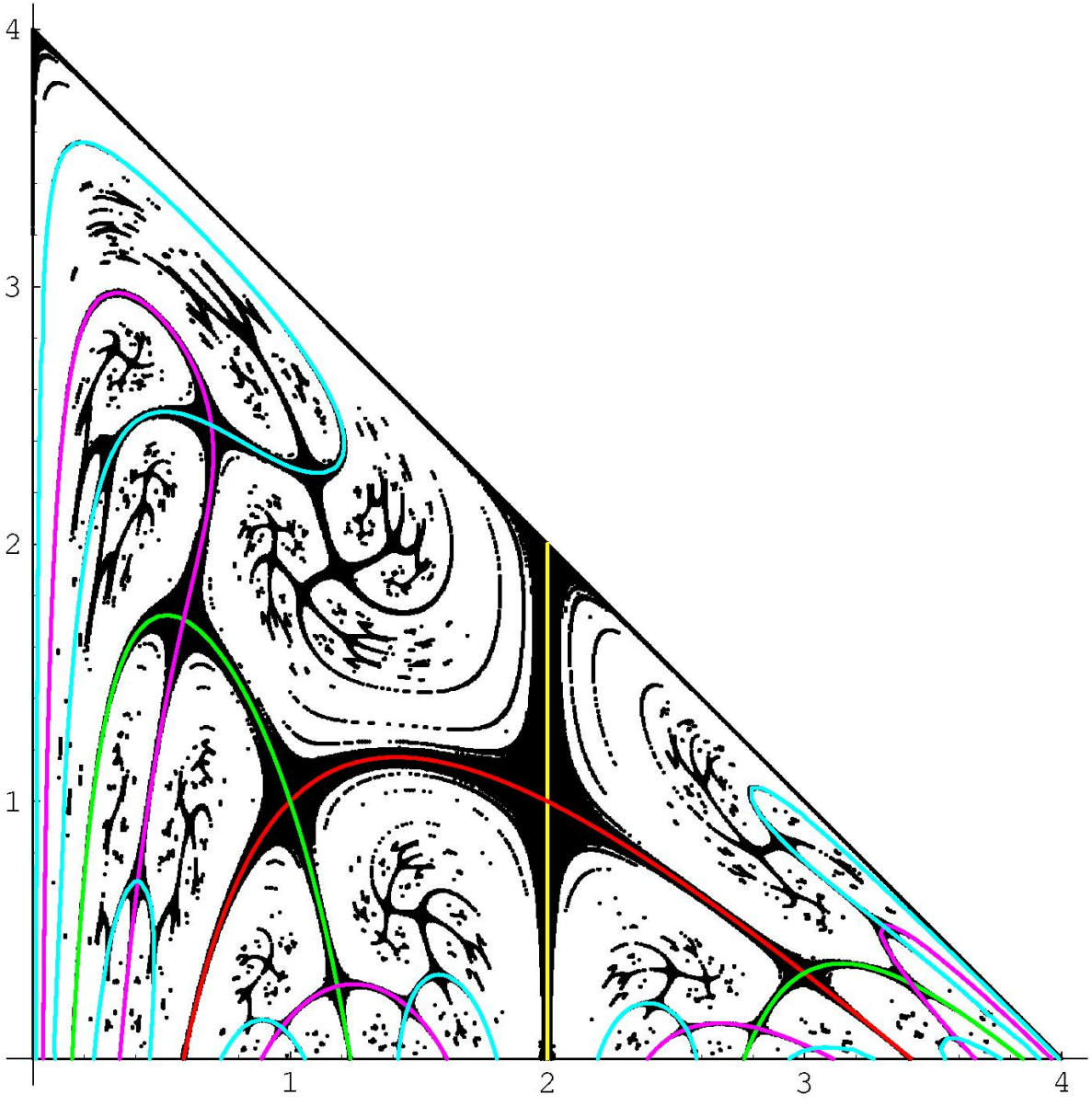
Interior points with period $n \leq 9$



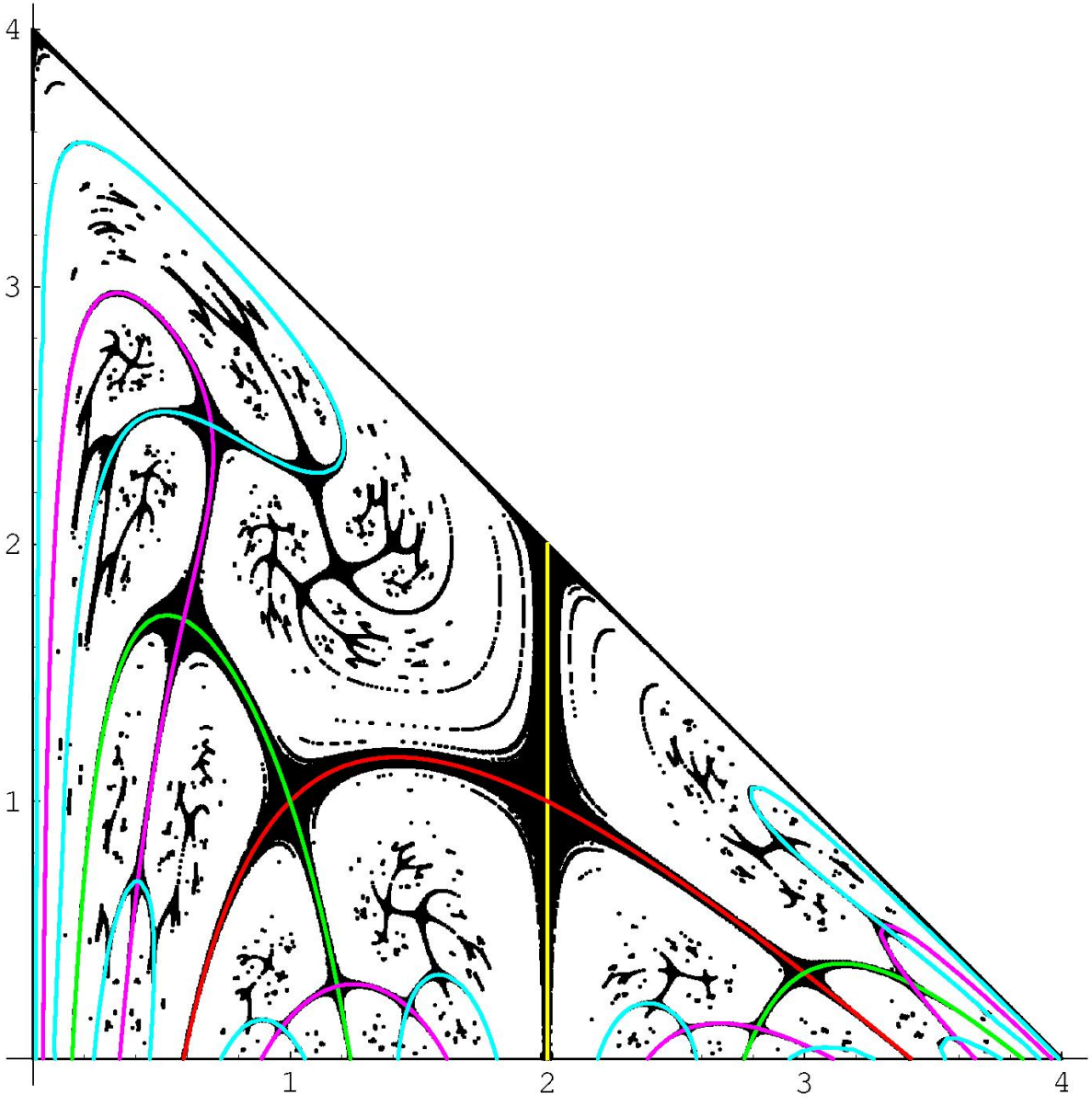
Interior points with period $n \leq 25$



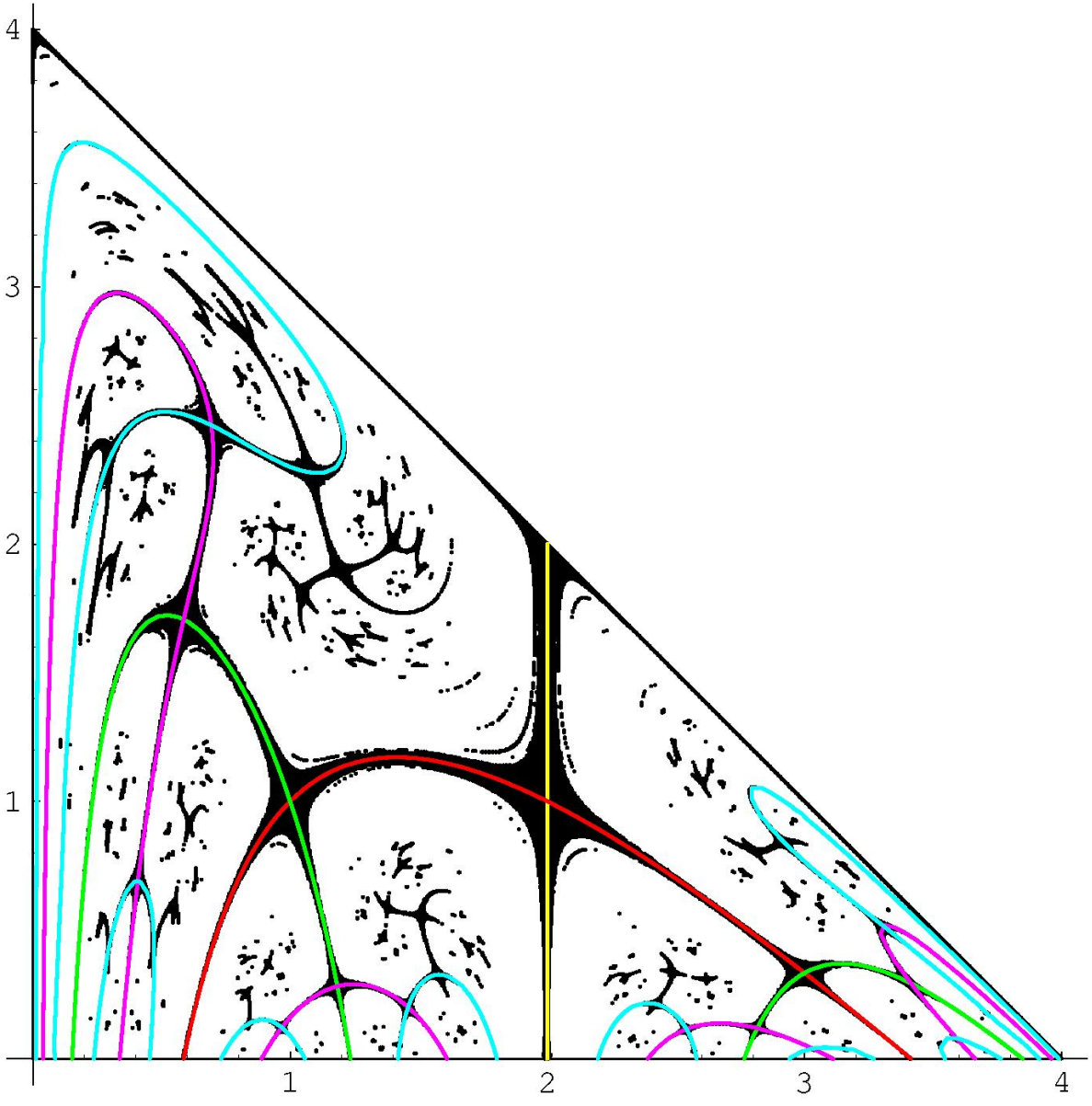
Interior points with period $n \leq 30$



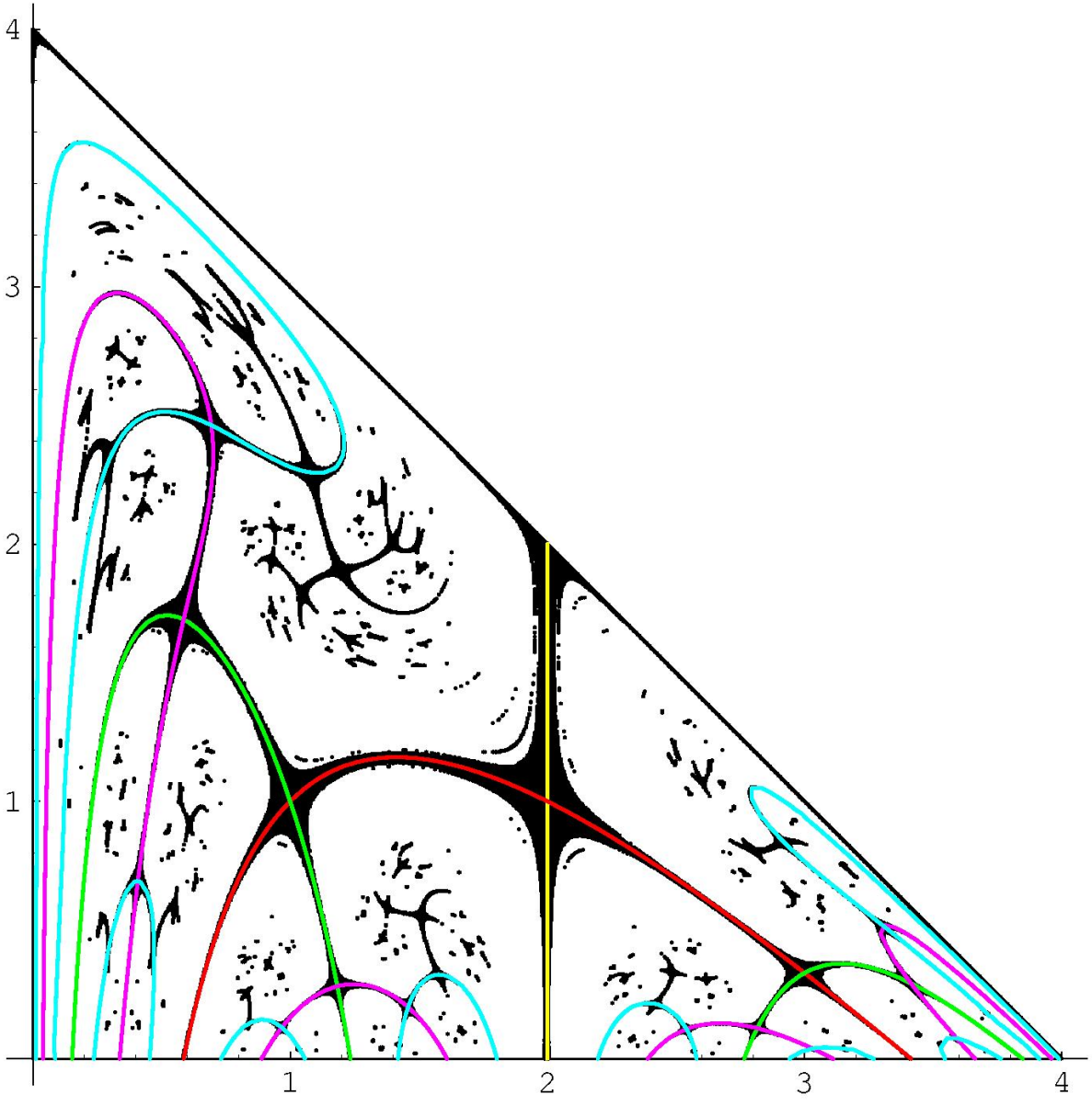
Interior points with period $n \leq 32$



Interior points with period $n \leq 35$



Interior points with period $n \leq 36$

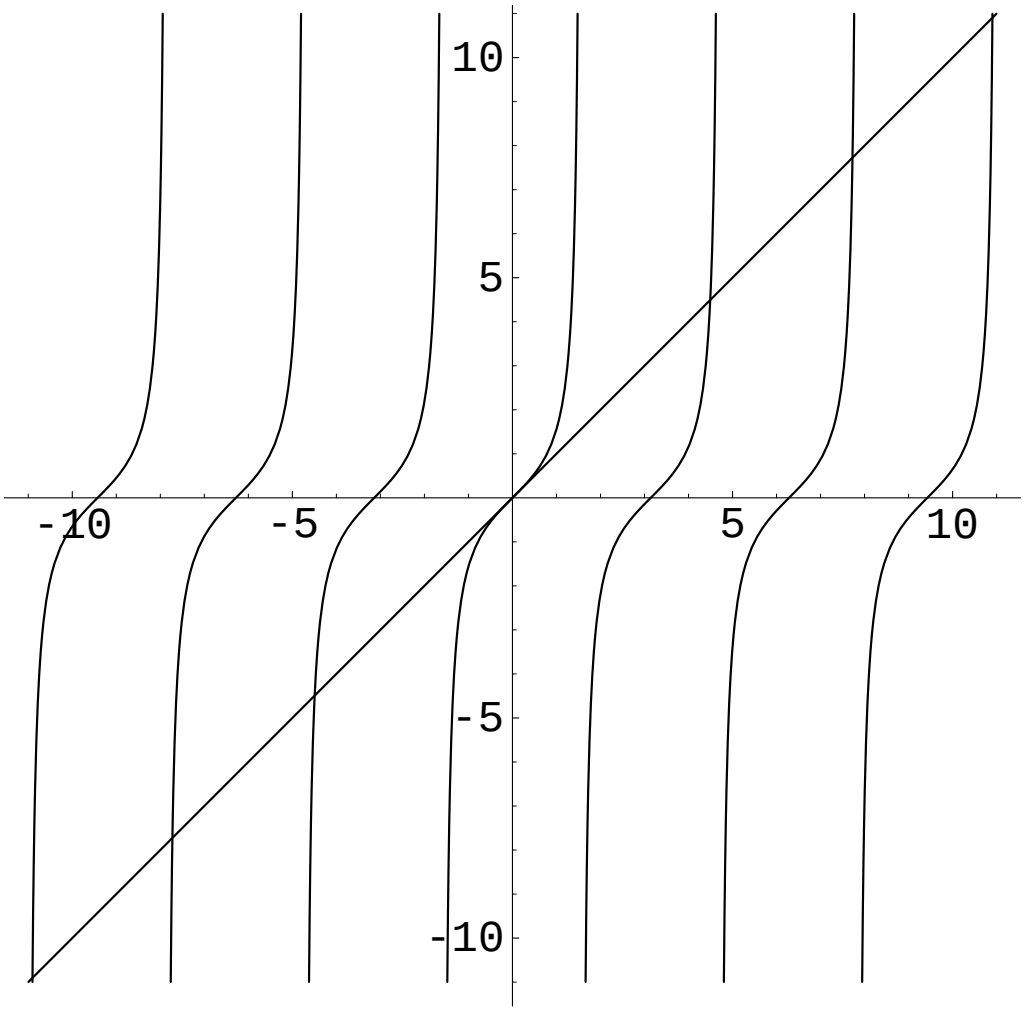


Attracting versus repulsive fixed points

If $x_0 = f(x_0)$ and $|f'(x_0)| < 1$ then

$$x_0 = \lim_{k \to \infty} x_k$$

where $x_k = f(x_{k-1})$ and x_1 is arbitrary but sufficiently closed to x_0.



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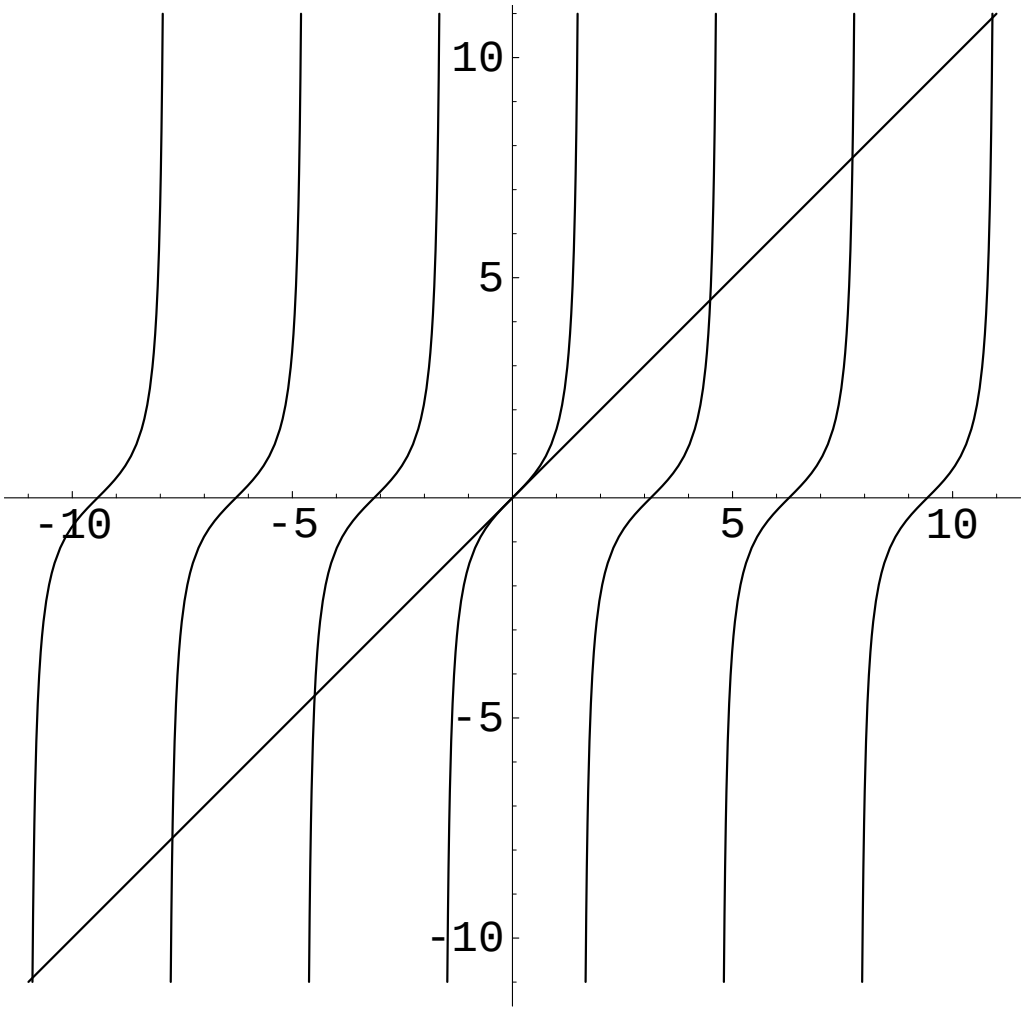


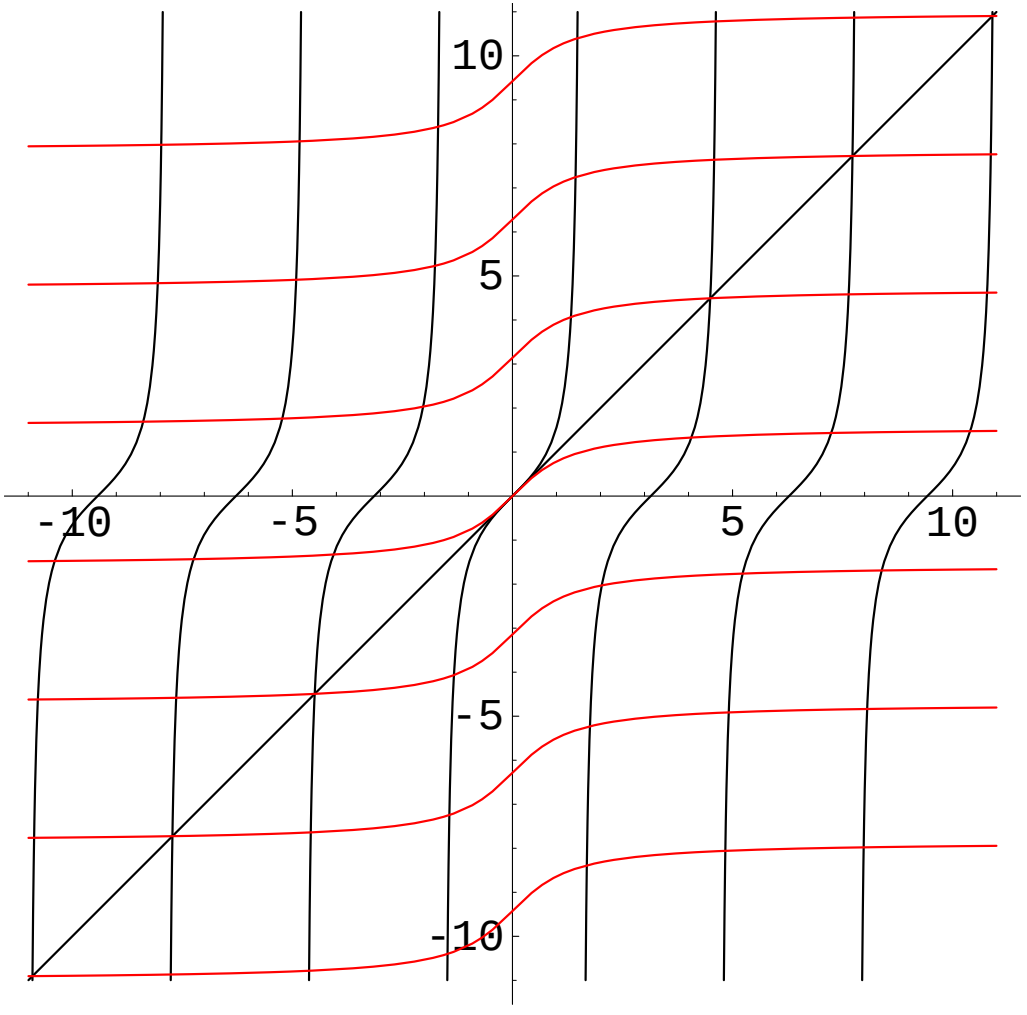
Attracting versus repulsive fixed points

The equation

$$x = \tan x$$

is equivalent to

$$x = k\pi + \arctan x\,, \text{ where } k \in \mathbb{Z}.$$



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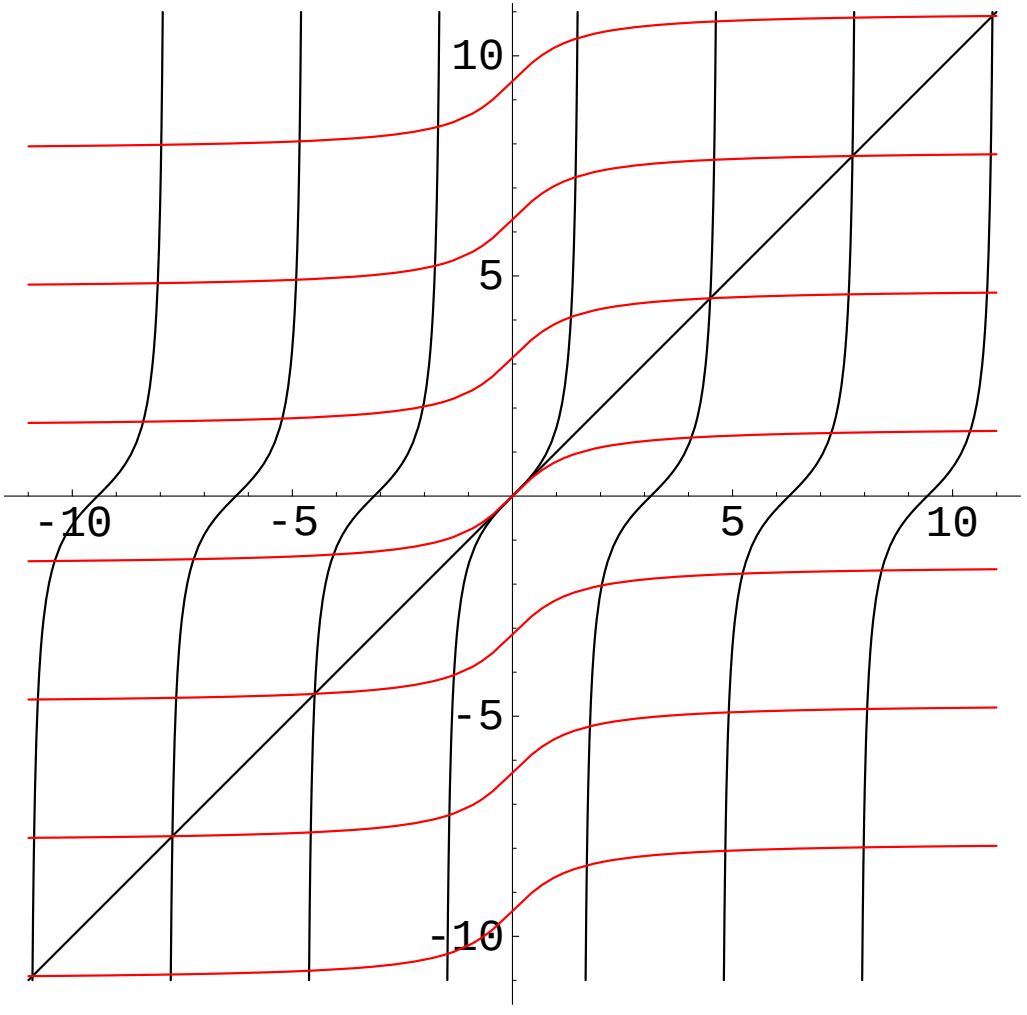
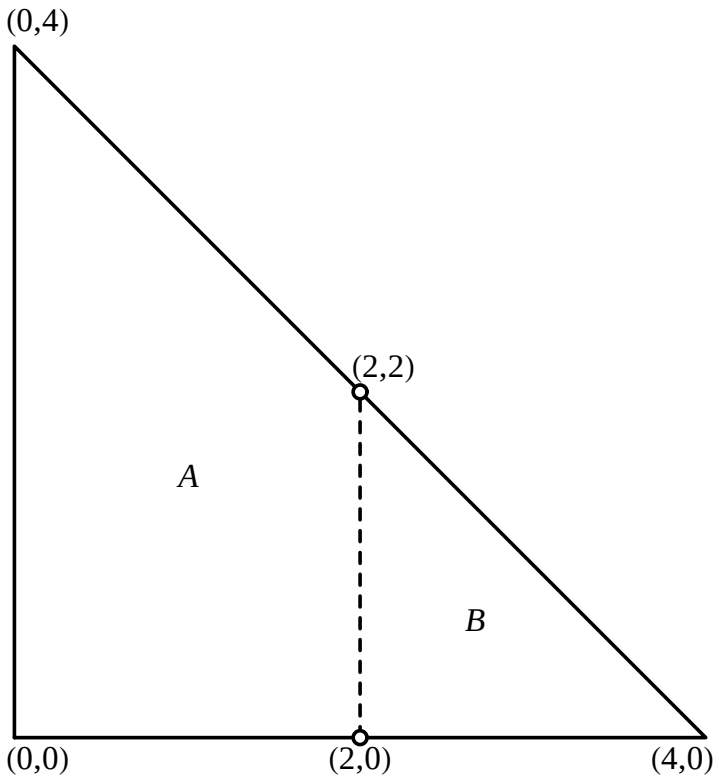


Relationship between lower and interior periodic points

Theorem (Main result)

Let P be a lower saddle fixed point of the map F^n. Then there is an interior fixed point Q of F^n with the same itinerary.



Itinerary

For a fixed point P of the map F^n it is sufficient to consider its itinerary W as a sequence $(w_i)_{i=0}^{n-1}$ defined by

$$w_i = \begin{cases} a & \text{if } F^i(P) \in A\,, \\ b & \text{if } F^i(P) \in B\,. \end{cases}$$

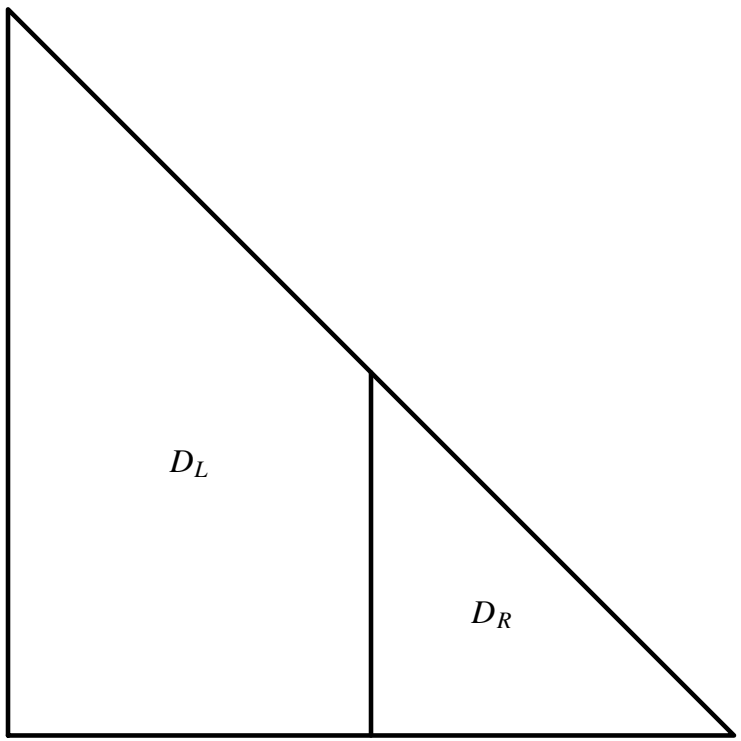
Such a sequence we will write in a shorten form

$$W = a^{j_1} b^{k_1} \dots a^{j_m} b^{k_m}\,.$$

Notation

It is natural to express the triangle D as the union

$$D = D_L \cup D_R \ ,$$



Notation

It is natural to express the triangle D as the union

$$D = D_L \cup D_R \ ,$$

where

$$D_L = \{\, [x, y] \in D : x \leq 2\} \ \text{ and}$$
$$D_R = \{\, [x, y] \in D : x \geq 2\} \ ,$$

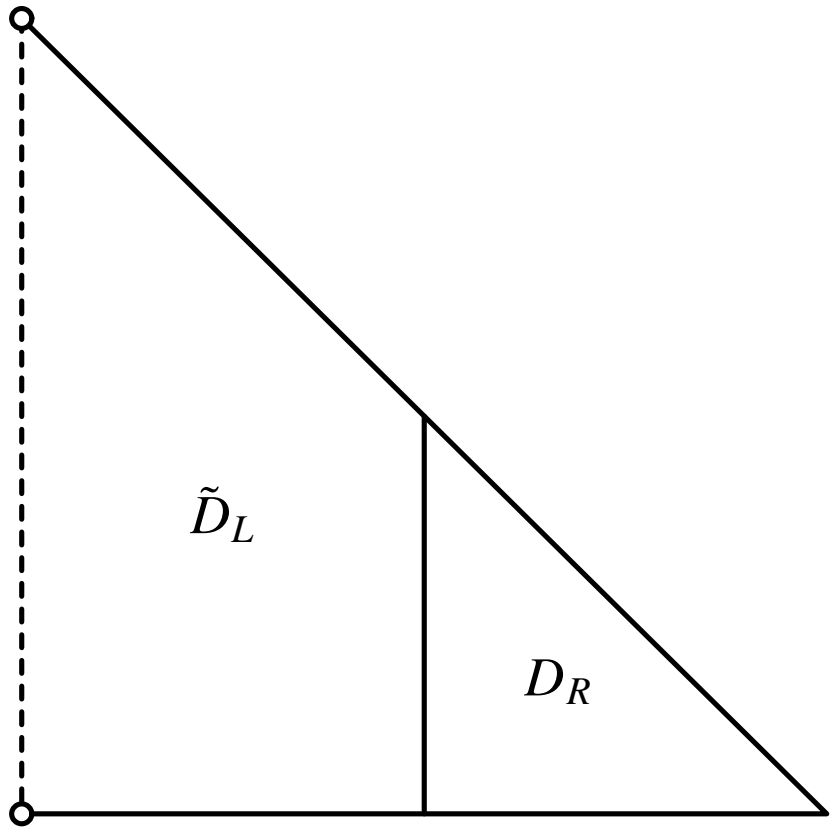
because

$$F(D_L) = D = F(D_R) \ .$$

Notation

Put also

$$\begin{aligned} \widetilde{D}_L &= \{\,[x,\,y] \in D : 0 < x \leq 2\} \text{ and} \\ \widetilde{D} &= D \setminus \{[0,0]\}. \end{aligned}$$



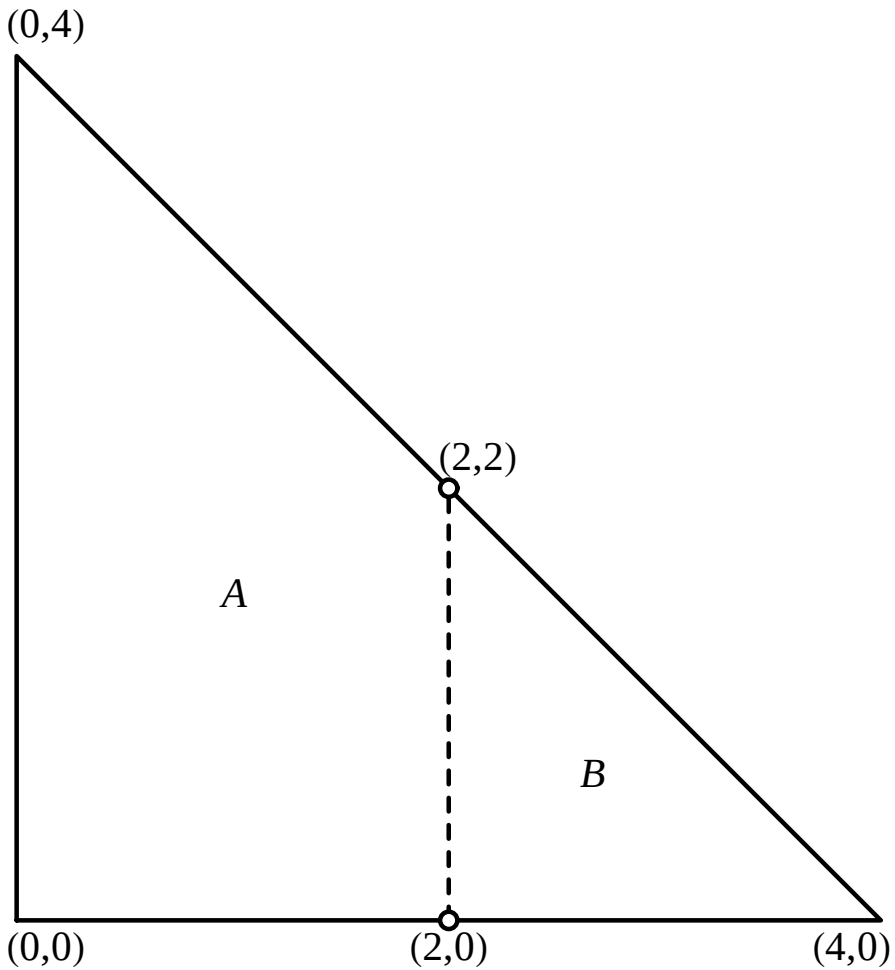
Inverse maps

The map F is not invertible, but F restricted to $\widetilde{D}_L$ and D_R is. The inverse maps of these restrictions are given by

$$L : \widetilde{D} \to \widetilde{D}_L\,, \quad [x,\, y] \mapsto \left[2 - \sqrt{4 - x - y},\ \frac{y}{2-\sqrt{4-x-y}}\right]$$

$$R : D \to D_R, \quad [x,\, y] \mapsto \left[2 + \sqrt{4 - x - y},\ \frac{y}{2+\sqrt{4-x-y}}\right]$$

Partition



Lower fixed point of F^n

Note that $F : [x, 0] \mapsto [f(x), 0]$, where
$f : \langle 0, 4\rangle \to \langle 0, 4\rangle$, $f(x) = x(4 - x)$ is the logistic map, which is conjugated with the tent map
$T : \langle 0, 1\rangle \to \langle 0, 1\rangle$, $T(t) = 1 - |1 - 2t|$

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Jacobi matrix

Let $P = [x_0, 0] \in D$ be a fixed point of the map F^n. In this case $P = \left[4 \sin^2 \frac{k\pi}{2^n \pm 1}, 0\right]$. Then the Jacobi matrix of the map F^n at the point P has a form

$$\begin{pmatrix} \lambda_1 & \mu \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \mp 2^n & \mu \\ 0 & \prod_{i=0}^{n-1} x_i \end{pmatrix},$$

where

$$[x_i, 0] = F^i(P) .$$

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Formula for λ_2

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Question

Take $n = 60$, sign $-$ and $k = 5124095576030431$.

$$\lambda_2 = ?$$

Classification

For λ_2 we have the possibilities

Saddle point

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Remark

All above points $[x_0, 0]$ have period 4.

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Main result

Theorem

Let P be a lower saddle fixed point of the map F^n. Then there is an interior fixed point Q of F^n with the same itinerary.

Explicit examples

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(i) If $P = [0, 0]$ then $Q = [1, 2]$ and $W = a$.

(ii) If $P = \left[4 \sin^2 \frac{\pi}{17}, 0\right]$ then $Q = \left[1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}\right]$ and $W = a^3 b$.

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(iii) If $P = \left[4\sin^2 \frac{\pi}{63}, 0\right]$ then $Q = \left[1, \frac{3+\sqrt{5}}{2}\right]$ and $W = a^4 b^2$.

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Sufficient condition for saddle point

Theorem

Let $P = \left[4\sin^2 \frac{k\pi}{2^n \pm 1}, 0\right]$, *where* n *and* k *are integers such that* $0 < n$ *and*

$$0 \leq k \leq \frac{\sqrt{2}(2^n \pm 1)}{\pi \cdot 2\sqrt{2n+1/4}} .$$

Then P *is a saddle fixed point of* F^n.

Notation

Let $\mathrm{IntFix}(F^n)$ be the set of all interior fixed points of the map F^n and $\mathrm{IntPer}(F, n)$ be the set all interior n-periodic of the map F.

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Estimates

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For cardinalities of $\mathrm{IntFix}(F^n)$ *and* $\mathrm{IntPer}(F, n)$ *we have the estimates*

1. $\#\,\mathrm{IntFix}(F^n) \geq \frac{2\sqrt{2}}{\pi} \cdot 2^{n-\sqrt{2n+1/4}}$

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1. $\#\,\mathrm{IntFix}(F^n) \geq \frac{2\sqrt{2}}{\pi} \cdot 2^{n-\sqrt{2n+1/4}}$
2. $\#\,\mathrm{IntPer}(F, n) \geq \frac{2\sqrt{2}}{\pi} \left(2^{n-\sqrt{2n+1/4}} - 2^{1+\frac{n}{2}}\right)$

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Let $W = a^{j_1} b^{k_1} \cdots a^{j_m} b^{k_m}$ *be an itinerary such that* $j_i > 0$, $k_i > 0$ *and* $\sum_{i=1}^{m} (j_i + k_i) = n$. *If*

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Particularly for $p \equiv 1 \pmod 8$ the set A_m contains at least two orbits.

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Particularly for $p \equiv 1 \pmod 8$ the set A_m contains at least two orbits.

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