Minimal sets in discrete dynamics

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Minimal sets in discrete dynamics

Minimal sets are the fundamental objects of study in topological dynamics.

D. V. Anosov, entry "Minimal set" in Kluwer's Encyclopaedia of mathematics

The classification of compact minimal sets in topological dynamics is a largely unsolved problem. Only for special classes something can be said. ... ... Unsolved is also the problem as to which (compact) Hausdorff spaces can be the phase space of a minimal flow or a minimal cascade.

"Expert Comments" added to the above

Minimal sets in discrete dynamics

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1. Minimal systems – basic facts

(X, f) .... X - topological space, $f : X \to X$ continuous
$x \in X$ .... (forward) **orbit** $\{x, f(x), f^2(x), \dots\}$

(X, f) .... **minimal** if there is no proper subset $M \subseteq X$ which is nonempty, closed and invariant (i.e. mapped **into** itself). In such a case we also say that f itself is minimal. (*Birkhoff 1912*)

Equivalent:

(1) (X, f) is minimal,
(2) every (forward) orbit is dense.

Minimal f is necessarily **surjective** if $X =$ compact Hausdorff.

1. Minimal systems – basic facts

Misunderstandings in case of homeomorphisms:

density of all forward orbits $\neq$ density of all full orbits
(however, in compact Hausdorff spaces this is the same)

Scottish book, Problem 115 (*Ulam (around 1930?)*): "$\exists$? homeo in $\mathbb{R}^2 \setminus \{\text{one point}\}$ such that all orbits are dense?"

- if forward orbits $\Longrightarrow$ "no" (easy, *Gottschalk 1944*)
- if full orbits $\Longrightarrow$ "no" (difficult, *LeCalvez, Yoccoz 1997*)

Remember: For us $\boxed{\text{orbit} = \text{forward orbit (even if } f \text{ is homeo)}}$

Equivalent (for X compact Hausdorff):

(1) (X, f) is minimal,

(2) $f(X) = X$ and every backward orbit ($=$ sequence) is dense.

(but "$\forall x \in X$, the full backward orbit $\bigcup_{n=0}^{\infty} f^{-n}(x)$ is dense" is not equivalent with minimality)

2. Simplest examples of minimal systems

- periodic orbit
- irrational rotation of the circle
- adding machine (odometer)
- Floyd-Auslander minimal system

These are homeos. However, there are examples of non-invertible minimal maps (on the Cantor set, on the torus, ...).

Many spaces do not admit any minimal map at all (nondegenerate spaces with fpp, ...).

3. Spaces admitting minimal maps

A **space** is said to be **minimal** if it admits a minimal map.

A zoo of non-minimal spaces:

- spaces with fpp (dendrites, disk, ...)
- spaces with ppp ($\mathbb{S}^{2n}$, ...)
- spaces satisfying the assumptions of Gottschalk's theorem: Theorem (*Gottschalk 1944*) *If X is a non-compact Hausdorff space with a compact subset having non-empty interior then X does not admit any minimal map.*
- continua having a cut point (for homeos *Erdös and Stone 1945*, for maps *Kelley 1947*)
- compact(!) metric spaces having countably infinite number of connected components
- and many concrete spaces (say Cantor set $\times$ $[0, 1]$, etc. etc.)

3. Spaces admitting minimal maps

A zoo of minimal spaces:

- finite sets, Cantor set, $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$, $\mathbb{T}^n$
- Klein bottle, $\mathbb{S}^1 \times \mathbb{S}^n$, ... (*Ellis 1965*, Klein bottle also by *Parry 1974*)
- $\mathbb{S}^{2n-1}$, ... (*Fathi and Herman 1977*)
- if an infinite compact metric space X is minimal then $X \times Z$ is minimal provided Z is "good" (e.g. any compact connected manifold without boundary is good) (*Glasner and Weiss 1979*, *Anosov and Katok, Fathi and Herman, Fathi, Fayad and Katok, Kolyada and Matviichuk, Dirbák and Maličký*)
- continua which are not locally connected (*Jones 1955*, ...)
- the pseudo-circle (*Handel 1982*)

3. Spaces admitting minimal maps

A zoo of minimal spaces (continuation):

- other nonhomogeneous continua, such as pinched torus, Sierpiński curve on the torus, pinched Sierpiński curve on the torus (*Bruin, Kolyada and S. 2003*)
- some disjoint unions of arcs and points (*Floyd 1949*, *Auslander 1988*)
- Menger universal curve (*Anderson, announced 1958*)
- compact metric spaces admitting a continuous minimal flow (*Fayad 2000*)

3. Spaces admitting minimal maps

In two important classes of spaces we are able to decide which of the spaces are minimal and which not.

- On 2-manifolds:

  Theorem (*Blokh, Oversteegen and Tymchatyn 2005*) *Among the 2-manifolds (compact or not, connected or not, with or without boundary) only the finite union of tori and the finite union of Klein bottles admit minimal maps.*

3. Spaces admitting minimal maps

- On almost totally disconnected compact metric spaces:

  Definition A compact metric space X is said to be **almost totally disconnected** if the set of its degenerate components, considered as a subset of X, is dense in X. A compact metric space X is said to be a **cantoroid** if it is almost totally disconnected and has no isolated point.

  Theorem (*Balibrea, Downarowicz, Hric, Špitalský and S. 2009*)
  An almost totally disconnected compact metric space admits a minimal map if and only if it is either a finite set or a cantoroid.

4. Existence of minimal sets

(X, f) .... $M \subseteq X$ **minimal set**, if $M \neq \emptyset$, closed, $f(M) \subseteq M$
and no proper subset of M has these three properties

In every **compact** system there are minimal sets.

5. Alternative "nowhere dense or the whole space"

On the circle, a minimal set is either nowhere dense or the whole circle. The same alternative **'nowhere dense or the whole space'** holds for minimal sets in the following cases:

- For **transitive maps** in compact spaces.
- For **homeomorphisms** in compact **connected** spaces.
- On **compact connected** 2-**manifolds**:

  Theorem (*Kolyada, Trofimchuk and S. 2008*) *Let $\mathcal{M}^2$ be a compact connected 2-dimensional manifold, with or without boundary, and let $f : \mathcal{M}^2 \to \mathcal{M}^2$ be a continuous map. If $M \subseteq \mathcal{M}^2$ is a minimal set of the dynamical system $(\mathcal{M}^2, f)$ then either $M = \mathcal{M}^2$ or M is nowhere dense in $\mathcal{M}^2$. (Moreover, if $M = \mathcal{M}^2$ then $\mathcal{M}^2$ is a 2-torus or a Klein bottle.)*

6. Topological structure of minimal sets

The full topological characterization of minimal sets is known on:

- compact **zero-dimensional** Hausdorff spaces (minimal sets = finite sets, Cantor sets)
- **graphs** (minimal sets = finite sets, Cantor sets, unions of finitely many pairwise disjoint circles)
- **local dendrites** (A dendrite is a locally connected continuum which contains no circle. A local dendrite is a continuum such that every its point has a neighbourhood whose closure is a dendrite.)

  Definition A **brain** is a cantoroid whose non-degenerate components are dendrites and form a null family. A **generalized brain** is a cantoroid whose non-degenerate components are local dendrites forming a null family and only finitely many of them contain circles.

6. Topological structure of minimal sets

Theorem (*Balibrea, Downarowicz, Hric, Špitalský and S. 2009*)
Let L be a local dendrite and let M be a subset of L. Then M is a minimal set for some dynamical system on L if and only if M is either a finite set or a finite union of disjoint circles or a generalized brain. (Moreover, every generalized brain can be embedded into a suitable local dendrite.)

Corollary:

- minimal sets on **dendrites** = finite sets and brains.

What is the structure of minimal sets of skew products?

Theorem (*Kolyada, Trofimchuk and S. 2009*) *Let (E, B, p, Γ) be a compact tree bundle, (E, F) and (B, f) dynamical systems with $p \circ F = f \circ p$ (i.e., F is fibre-preserving). Then F has only nowhere dense minimal sets.*

Not true in graph bundles.

Corollary:

- Let $F(x, y) = (f(x), g(x, y))$ be a continuous triangular map in the square I^2 and let M be a minimal set of F. Then M is nowhere dense in the space $\mathrm{pr}_1(M) \times I$.

Other results in progress.

7. Open problems

Problem. The pseudo-circle admits a minimal homeo. Does it admit a minimal non-invertible map?

Conjecture. An n-manifold ($n \geq 2$) admits a minimal homeomorphism if and only if it admits a minimal non-invertible map.

Conjecture (suggested by J. Auslander). No non-degenerate non-separating plane continuum admits a minimal map.

Problem. Prove or disprove the alternative 'nowhere dense or the whole space' for minimal sets on compact connected n-dimensional manifolds, $n \geq 3$.

Problem. Characterize minimal sets of triangular maps in the square.

8. Some tools for the study of minimality

General properties of minimal dynamical systems. If (X, f) is minimal with X compact metric, then in many aspects f behaves like a homeo (*Kolyada, Trofimchuk and S. 2001*):

- there is **no redundant open set for** f (a set $G \subseteq X$ is said to be a redundant open set for $f : X \to X$ if G is nonempty, open and $f(G) \subseteq f(X \setminus G)$)
- f **is feebly open** (sends nonempty open sets to sets with non-empty interior)
- f **preserves the topological size of a set** in both directions: $A \subseteq X$ is nowhere dense (dense, 1st cat., 2nd cat., residual, has Baire property, has nonempty interior) $\Rightarrow$ so are both $f(A)$ and $f^{-1}(A)$
- f **is almost 1-to-1**: $\{x \in X : \mathrm{card}\, f^{-1}(x) = 1\}$ is G_δ-dense in X

8. Some tools for the study of minimality

Tools for the study of minimality on 2-manifolds:

- **Monotone-light factorization:**
  $f : \mathcal{M}^2 \to \mathcal{M}^2; \quad f = g \circ m$
  $m : \mathcal{M}^2 \to K$ monotone (cont., point inverses connected)
  $g : K \to \mathcal{M}^2$ light (cont., point inverses totally disconn.)
- **Monotone images of compact connected 2-manifolds without boundary are known** (*Roberts, Steenrod 1938*):

  $K =$ a compact metric space which can be obtained from a **generalized cactoid** by performing, consecutively, finitely many (possibly zero) times the operation of the identification of just two points.
- **Some properties of light and almost 1-to-1 maps** (*B*lokh, Oversteegen, Tymchatyn 2006).
- Etc.

8. Some tools for the study of minimality

Tools for the construction of a minimal map Ψ on a given cantoroid X:

- a system (X, Ψ) on the cantoroid X which is a strongly **almost 1-to-1 extension** of a Cantor minimal system is minimal
- such a map Ψ is obtained as a uniform limit of maps which are 'more and more continuous'
- Etc.

8. Some tools for the study of minimality

Tools for the study of the structure of a minimal set M in a graph bundle:

- we define the notion of a **strongly star-like interior point** of M (no connection with interior points of M)
- we carefully study possible neighbourhoods of compact (connected) sets which are subsets of a fibre and consist of strongly star-like interior points of M
- we exclude many 'shapes' of a minimal set M by showing that such a shape implies the existence of a redundant open set for the minimal map $F|_M$
- Etc.