# Transformations in Data Processing (Data Transformation and Modeling 1/2)

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# Content

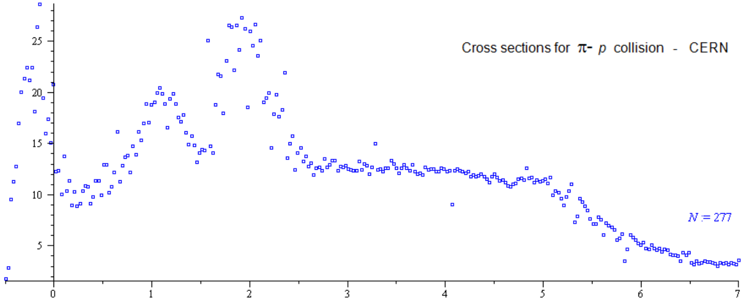
# Key terms

- reference points
- $r$-point transformation of functions
- IZA reprezentation of polynomials
- reparameterization
- (parametric estimating)

# Key terms

- reference points: $\mathcal{R} = \{(x_0, y_0), (x_1, y_1), \dots (x_{r-1}, y_{r-1})\}$
- $r$-point transformation of functions: $T_{\mathcal{R}} f(x)$
- IZA reprezentation of polynomials: $P(x) = I(x) + Z(x)A(x)$
- reparameterization: $(a_0, a_1, \dots, a_r, \dots, a_p) \Rightarrow \mathcal{R} + (a_r, \dots, a_p)$
- (parametric estimating)

# Our Motivation



# Our Motivation

# $r$-point transformation of functions

## Definition

The forward $r$-point transformation, $r-1 \in Z^+$, of any continuous function $f(x)$ based on a set of $r$ reference points $\mathcal{R} = \{[x_i, y_i],\ y_i = f(x_i),\ i = \overline{0, r-1}\}$ is given by

$$T_{\mathcal{R}} f(x) \equiv T_r f(x) = H_0(x) f(x) + \sum_{i=1}^{r-1} H_i(x) y_i,$$

where

$$H_i(x) = \prod_{v \in V_i} \frac{x_0 - v}{v_i - v}, \quad V_i = V \backslash \{v_i\}, V = \{x, x_1, \ldots, x_{r-1}\},\ i = \overline{0, r-1},$$

$$x_0 \neq x_1 \neq \ldots \neq x_{r-1} \text{ and } x \neq x_i,\ i = \overline{1, r-1}.$$

# $r$-point transformation of functions

Explicitly:

$$
\begin{aligned}
T_r f(x) &= \overbrace{\frac{(x_0-x_1)(x_0-x_2)\dots(x_0-x_{r-1})}{(x-x_1)(x-x_2)\dots(x-x_{r-1})}}^{H_0(x)} f(x) + \\
&+ \underbrace{\frac{(x_0-x)(x_0-x_2)\dots(x_0-x_{r-1})}{(x_1-x)(x_1-x_2)\dots(x_1-x_{r-1})}}_{H_1(x)} y_1 + \dots + \underbrace{\frac{(x_0-x)(x_0-x_1)\dots(x_0-x_{r-2})}{(x_{r-1}-x)(x_{r-1}-x_1)\dots(x_{r-1}-x_{r-2})}}_{H_{r-1}(x)} y_{r-1}.
\end{aligned}
$$

The simplest case, $r=2$, $\{[x_0,y_0],\ [x_1,y_1],\ y_i=f(x_i),\ i=0,1\}$ :

$$
\begin{aligned}
T_2 f(x) &= \frac{x_0-x_1}{x-x_1} f(x) + \frac{x_0-x}{x_1-x} y_1, \\
T_2^{-1} T_2 f(x) &= \frac{x-x_1}{x_0-x_1} T_2 f(x) + \frac{x-x_0}{x_1-x_0} y_1, \\
l_1(x) &= \frac{x-x_1}{x_0-x_1} y_0 + \frac{x-x_0}{x_1-x_0} y_1.
\end{aligned}
$$

# $r$-point transformation

Properies of the transformation:

- polynomial degree reduction
- linearicity

Examples:

$$\begin{aligned}
T_2 x^4 &= x_0{}^4 + (x_0 - x_1)(x - x_0)\left(x^2 + (x_0 + x_1)x + x_0{}^2 + x_1(x_0 + x_1)\right) \\
T_3 x^3 &= x_0{}^3 + (x_0 - x_1)(x_0 - x_2)(x - x_0) \\
T_4 x^5 &= x_0{}^5 + (x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x - x_0)(x + x_0 + x_1 + x_2 + x_3)
\end{aligned}$$

$$\begin{aligned}
T_2 P_4(x) &= P_4(x_0) + (x_0 - x_1)(x - x_0)\,A_{2,2} \\
T_3 P_5(x) &= P_5(x_0) + (x_0 - x_1)(x_0 - x_2)(x - x_0)\,A_{2,3}
\end{aligned}$$

# IZA reprezentation of polynomials: P = I + ZA

## Theorem

*Assume* $p \geq r \geq 2$. *Then any polynomial* $P_p(x)$ *can be expressed based on its any different r points* $\{[x_i,\ y_i],\ y_i = P_p(x_i),\ i = \overline{0, \mathrm{r}-1}\}$ *as*

$$P_p(x) = I_{r-1}(x) + Z_r(x)A_{p-r,r}(x),$$

- $I_{r-1}(x) = \sum_{i=0}^{r-1} \Pi_i(x)y_i$ *is an incomplete interpolating polynomial,*

$\Pi_i(x) = \prod_{v \in V_i} \frac{x-v}{x_i-v}, \quad V_i = V \backslash \{v_i\},\ \ V = \{x_0, x_1, \ldots, x_{r-1}\},\ \ i = \overline{0, r-1}.$

- $Z_r(x) = \prod_{i=0}^{r-1}(x - x_i)$, - $A_{p-r,r}(x) = \mathbf{S}^T \cdot \alpha,$

$\mathbf{S} = (S_{0,r},\ S_{1,r} \ldots,\ S_{p-r,r})^T,\ \alpha = (a_r,\ a_{r+1} \ldots,\ a_p)^T,\ S_{1,r} = 1,\ r \geq 0,$
$S_{j,0} = x_0^j,\ j \geq 0, \quad S_{j,r} = S_{j,r-1} + x_r S_{j-1,r}, \quad j \geq 1,\ r \geq 1.$

# IZA reprezentation of polynomials: P = I + ZA

IZA$(p, r)$:

$$\begin{aligned} P_p(x) &= I_{r-1}(x) + Z_r(x)\ A_{p-r,r}(x), \\ &= I_{r-1}(x) + Z_r(x)\ \mathbf{S}^T \cdot \alpha. \end{aligned}$$

$S_{1,r} = 1,\ r \geq 0,\ S_{j,0} = x_0^j,\ j \geq 0,$

$$S_{j,r} = S_{j,r-1} + x_r S_{j-1,r}, \quad j \geq 1,\ r \geq 1.$$

$S_{2,1} = S_{2,0} + x_1 S_{1,1} = x_0^2 + x_1(x_0 + x_1),$

$$\begin{aligned} S_{3,2} &= S_{3,1} + x_2 S_{2,2} = (S_{3,0} + x_1 S_{2,1}) + x_2(S_{2,1} + x_2 S_{1,2}) = \\ &= x_0^3 + x_1(x_0^2 + x_1(x_0 + x_1)) + x(x_0^2 + x_1(x_0 + x_1) + x(x_0 + x_1 + x)). \end{aligned}$$

# IZA reprezentation of polynomials: P = I + ZA

IZA($p$, $r$) examples:

`IZA(4,2):`

$$P_4(x) = \frac{(x - x_1)\,P_4(x_0)}{x_0 - x_1} + \frac{(x - x_0)\,P_4(x_1)}{x_1 - x_0} + (x - x_0)\,(x - x_1)$$
$$\left(a_2 + a_3\,(x + x_0 + x_1) + a_4\left(x^2 + (x_0 + x_1)\,x + {x_0}^2 + x_1\,(x_0 + x_1)\right)\right).$$

`IZA(3,3):`

$$P_3(x) = \frac{(x - x_1)\,(x - x_2)\,P_3(x_0)}{(x_0 - x_1)\,(x_0 - x_2)} + \frac{(x - x_0)\,(x - x_2)\,P_3(x_1)}{(x_1 - x_0)\,(x_1 - x_2)}$$
$$+ \frac{(x - x_0)\,(x - x_1)\,P_3(x_2)}{(x_2 - x_0)\,(x_2 - x_1)} + (x - x_0)\,(x - x_1)\,(x - x_2)\,a_3.$$

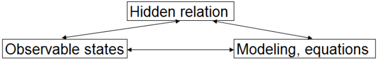
# IZA reprezentation of polynomials: P = I + ZA

## reparametrization - two types of parameters

$p = 6$, $r = 2$, $x_0 \neq x_1$, $\alpha = (a_2, a_3, \ldots, a_6)$

$$P_6(x) = \overbrace{\frac{(x - x_1)P_6(x_0)}{x_0 - x_1} + \frac{(x - x_0)P_6(x_1)}{x_1 - x_0}} + \overbrace{(x - x_0)(x - x_1)}[a_2 + (x + x_0 + x_1)a_3 +$$
$$+(x^2 + (x_0 + x_1)x + x_0{}^2 + x_1(x_0 + x_1))a_4 +$$
$$+(x^3 + (x_0 + x_1)x^2 + (x_0{}^2 + x_1(x_0 + x_1))x + x_0{}^3 + x_1(x_0{}^2 + x_1(v_0 + x_1)))a_5 +$$
$$+(x^4 + (x_0 + x_1)x^3 + (x_0{}^2 + x_1(x_0 + x_1))x^2 + (x_0{}^3 + x_1(x_0{}^2 + x_1(x_0 + x_1)))x + x_0{}^4 + x_1(x_0{}^3 + x_1(x_0{}^2 + x_1(x_0 + x_1))))a_6]$$

# Summary




**relation = states/interpolation + equation/approximation.**

- IZA reprezentation - **reparameterization.**
- IZA reprezentation expresses the relation by explicit use of several observed **states** as reference points.
- IZA reprezentation can potentially leverage the **advantages** and avoid the drawbacks of interpolation and approximation.
- IZA reprezentation can express by appropriate localization of the reference points the connection between **neighboring** relations as smooth transition between two local approximants.