

Perfect matchings in cubic graphs

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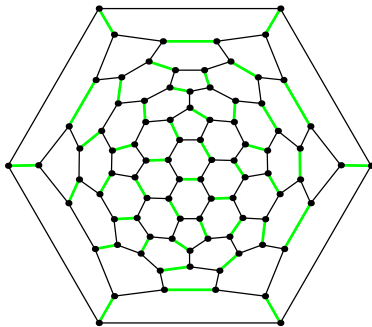
Nový Smokovec, June 21-23, 2010



Podporujeme výskumné aktivity na Slovensku/
Projekt je spolufinancovaný zo zdrojov EÚ



What is it?

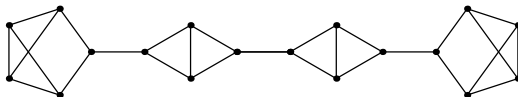


Conjecture (Lovász & Plummer 70's)

There exists a constant $c > 0$, such that any n -vertex cubic bridgeless graph contains at least 2^{cn} perfect matchings.

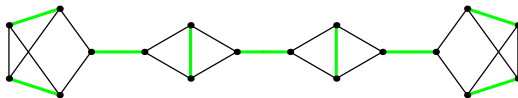
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Let G be a bipartite cubic bridgeless graph with n vertices.

- $m(G) \geq n/2$ (Sinkhorn 1969)
- $m(G) \geq n/2 + 2$ (Minc 1969)
- $m(G) \geq 3n/2 - 3$ (Hartfiel & Crosby 1971)
- $m(G) \geq 6 \cdot (4/3)^{n/2-3}$ (Voorhoeve 1979)

Theorem (Schrijver 1998)

Every bipartite regular bridgeless graph contains an exponential number of perfect matchings.

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Theorem (Schrijver 1998)

Every bipartite regular bridgeless graph contains an exponential number of perfect matchings.

Theorem (Chudnovsky & Seymour 2008)

Every planar cubic bridgeless graph with n vertices contains at least $2^{n/655978752}$ perfect matchings.

Theorem (Edmonds, Lovász & Pulleyblank; Naddef 1982)

Every cubic bridgeless graph with n vertices contains at least $n/4 + 2$ perfect matchings.

Theorem (Král', Sereni & Stiebitz 2008)

Every cubic bridgeless graph with n vertices contains at least $n/2$ perfect matchings.

Theorem (Esperet, Král', Škoda & Škrekovski 2008)

Every cubic bridgeless graph with n vertices contains at least $3n/4 - 10$ perfect matchings.

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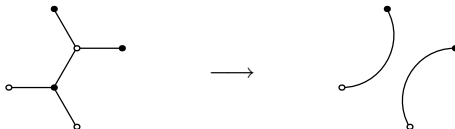
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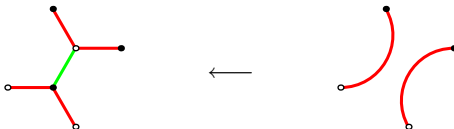
Theorem (Esperet, K. & Král' 2009)

For any $a > 0$ there exists a constant b such that every cubic bridgeless graph with n vertices contains at least $an - b$ perfect matchings.

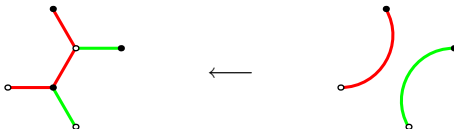
Splitting an edge



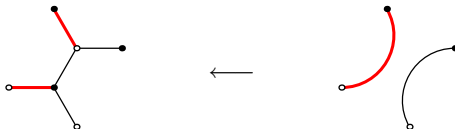
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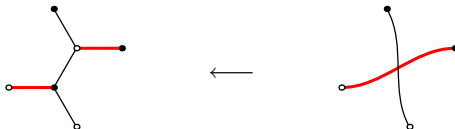
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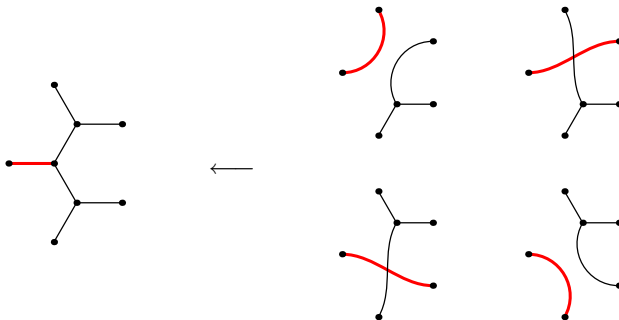
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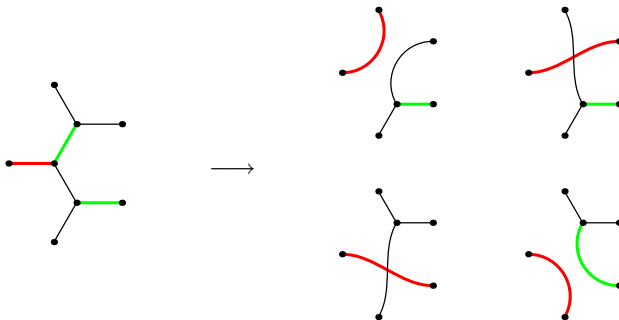
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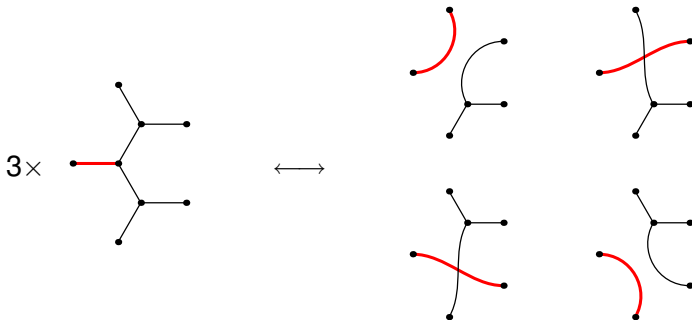
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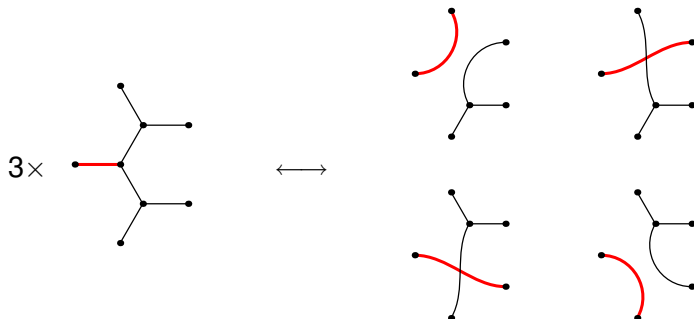
$$m_e(n) \geq \frac{4}{3} \cdot m_e(n-2).$$

Theorem (Voorhoeve 1979)

Let G be a bipartite cubic graph with n vertices. Then

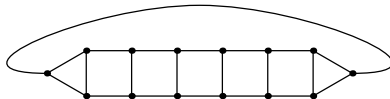
$$m(G) \geq 6 \cdot (4/3)^{n/2-3}.$$

Troubles in the general case

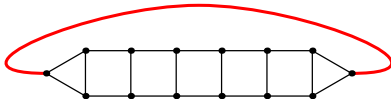


In general, the number of perfect matchings avoiding a given edge does not grow exponentially:

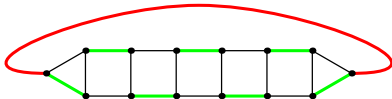
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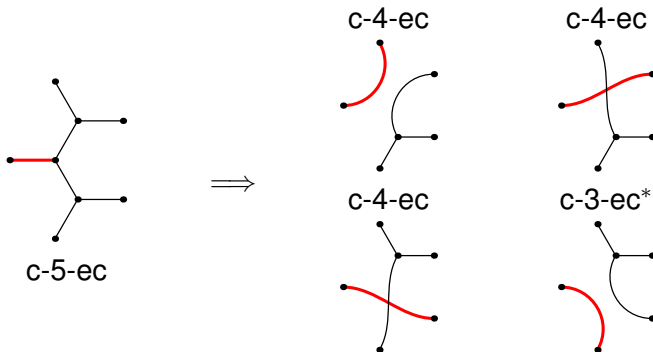
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Cyclically 5-edge-connected graphs



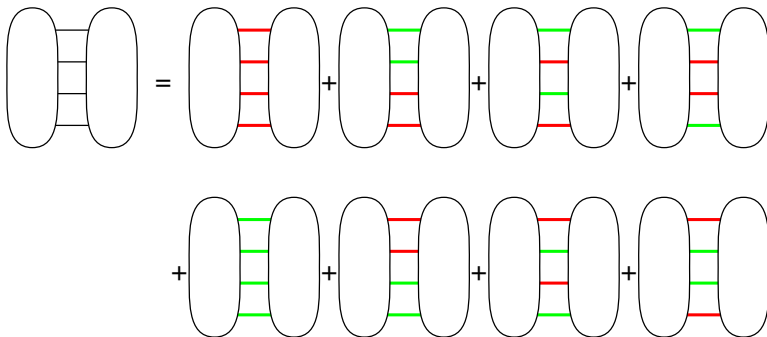
c- k -ec: a cyclically k -edge-connected graph

c- k -ec*: a cyclically k -edge-connected graph, with no cyclic k -edge cut containing the forbidden edge e

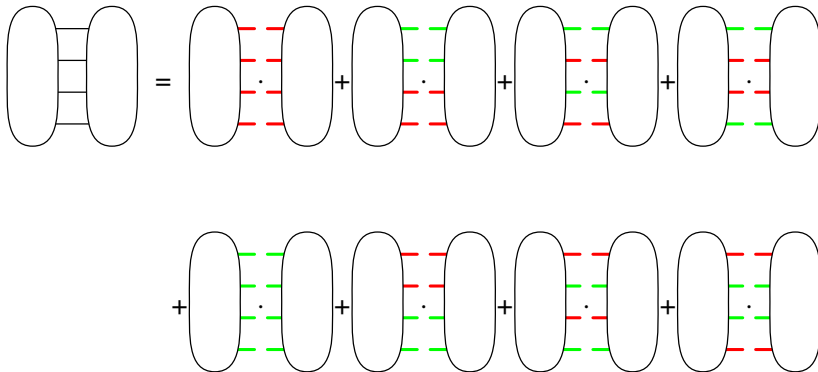
Lemma

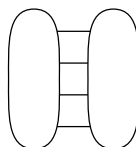
Let G be an n -vertex cyclically 3-edge-connected cubic graph and e an edge of G that is not contained in any cyclic 3-edge-cut of G . The number of perfect matchings of G that avoids e is at least $n/8$.

Graphs with cyclic 4-edge-cuts



Graphs with cyclic 4-edge-cuts





$$= \sum_{i=1}^8 x_i y_i$$

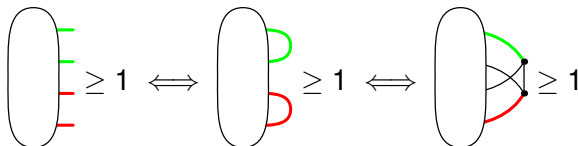
If $x_i \geq 1$ and $y_i \geq 1$ for some $i \in I \subseteq \{1, 2, \dots, 8\}$, then

$$x_i y_i \geq x_i + y_i - 1$$

and thus

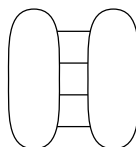
$$\sum_{i \in I} x_i y_i \geq \sum_{i \in I} x_i + \sum_{i \in I} y_i - 8.$$

Graphs with cyclic 4-edge-cuts



Lemma (Esperet, Král', Škoda & Škrekovski 2008)

Let G be a cyclically 4-edge-connected cubic graph and e and f two edges of G . G contains no perfect matchings avoiding e and containing f if and only if the graph $G \setminus \{e, f\}$ is bipartite and the end-vertices of e are in one color class while and the end-vertices of f are in the other.



$$= \sum_{i=1}^8 x_i y_i$$

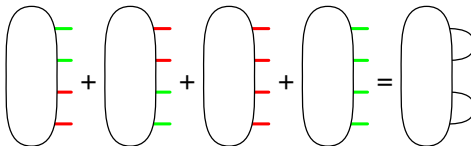
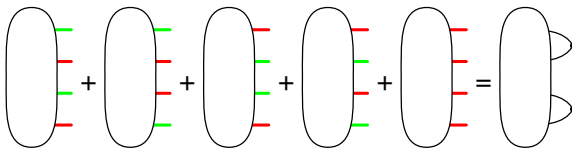
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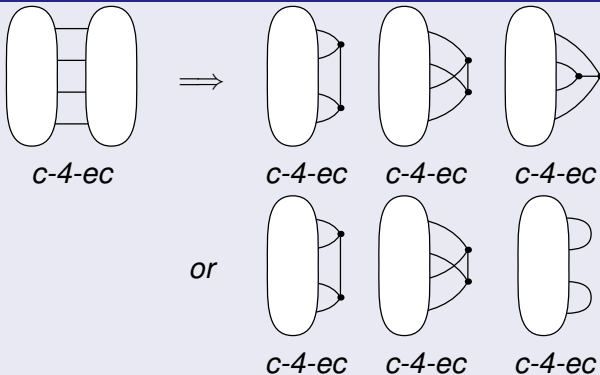
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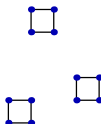
Graphs with cyclic 4-edge-cuts



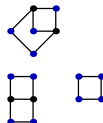
Lemma



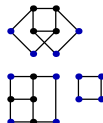
A *twisted net* is either a 4-cycle, or the graph inductively obtained from a twisted net G and a twisted net or a single edge H by adding edges uv and $u'v'$ to the disjoint union of G and H , where u, u' and v, v' are vertices of degree less than 3 in G and H , respectively.



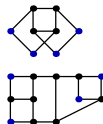
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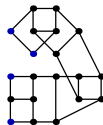
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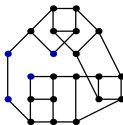
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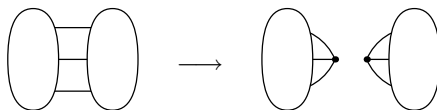


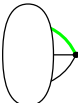
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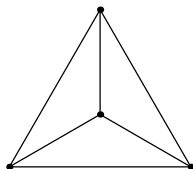


If all  ≥ 2 , we apply induction.

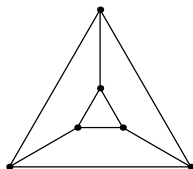
Lemma (Esperet, Král', Škoda & Škrekovski 2008)

Every cyclically 3-edge-connected cubic graph that is not a Klee-graph is double covered.

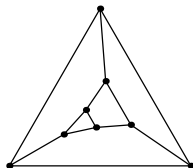
A *Klee-graph* is either K_4 , or the graph obtained from a Klee-graph by replacing a vertex by a triangle.



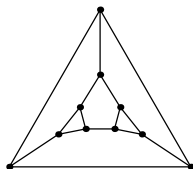
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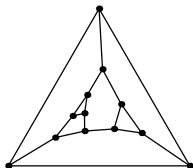
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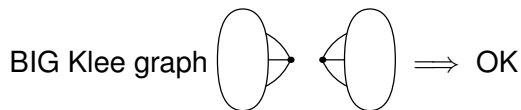


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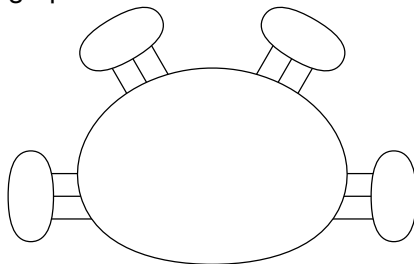


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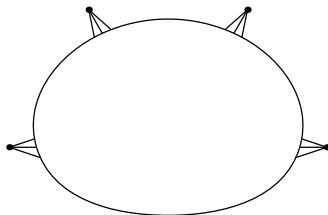




only small Klee graphs



only small Klee graphs



\Rightarrow c-4-ec graph



Thank you for your attention!