Autonomous automata

versus

monounary algebras

Danica Jakubíková-Studenovská
P.J.Šafárik University, Košice, Slovakia

Pracovné stretnutie riešitel'ov Centra excelentnosti CEX CaKS
Nový Smokovec
21.6. – 23.6.2010

Európska únia
Európsky fond regionálneho rozvoja
Operačný program VÝSKUM a VÝVOJ
Agentúra Ministerstva školstva SR pre štrukturálne fondy EÚ
Podporujeme výskumné aktivity na Slovensku/
Projekt je spolufinancovaný zo zdrojov EÚ

Introduction

Introduction

- second half of the last century: a development and a growth of the meaning of the concepts

Introduction

- second half of the last century: a development and a growth of the meaning of the concepts
  - information

Introduction

- second half of the last century: a development and a growth of the meaning of the concepts
    - information
    - transmission of information

Introduction

- second half of the last century: a development and a growth of the meaning of the concepts
  - information
  - transmission of information

- **automaton** is a system, which accepts signals from its environment and according to the signals it changes its internal state and transmits signals to the environment

Introduction

- second half of the last century: a development and a growth of the meaning of the concepts
  - information
  - transmission of information

- **automaton** is a system, which accepts signals from its environment and according to the signals it changes its internal state and transmits signals to the environment
- avoid too complicated automata with various special properties

Introduction

- second half of the last century: a development and a growth of the meaning of the concepts
  - information
  - transmission of information

- **automaton** is a system, which accepts signals from its environment and according to the signals it changes its internal state and transmits signals to the environment
- avoid too complicated automata with various special properties
- give a more general approach

Introduction

- second half of the last century: a development and a growth of the meaning of the concepts
    - information
    - transmission of information

- **automaton** is a system, which accepts signals from its environment and according to the signals it changes its internal state and transmits signals to the environment
- avoid too complicated automata with various special properties
- give a more general approach
    - **abstraction**

Introduction

- second half of the last century: a development and a growth of the meaning of the concepts
  - information
  - transmission of information

- **automaton** is a system, which accepts signals from its environment and according to the signals it changes its internal state and transmits signals to the environment
- avoid too complicated automata with various special properties
- give a more general approach
  - **abstraction**
  - generalize the concept

Introduction

- second half of the last century: a development and a growth of the meaning of the concepts
  - information
  - transmission of information

- **automaton** is a system, which accepts signals from its environment and according to the signals it changes its internal state and transmits signals to the environment
- avoid too complicated automata with various special properties
- give a more general approach
  - **abstraction**
  - generalize the concept
  - apply scientific exact mathematical tools

Introduction

Introduction

- S.Ginsburg: An introduction to mathematical machine theory, 1962

Introduction

- S.Ginsburg: An introduction to mathematical machine theory, 1962
- F.Gécseg, I.Peák: Algebraic theory of automata, 1972

Introduction

- S.Ginsburg: An introduction to mathematical machine theory, 1962
- F.Gécseg, I.Peák: Algebraic theory of automata, 1972
- V.N.Salij: Universal algebra and automata, in Russian, 1988

Introduction

- S.Ginsburg: An introduction to mathematical machine theory, 1962
- F.Gécseg, I.Peák: Algebraic theory of automata, 1972
- V.N.Salij: Universal algebra and automata, in Russian, 1988
- J.M.Hovie: Automata and languages, 1991

Introduction

- S.Ginsburg: An introduction to mathematical machine theory, 1962
- F.Gécseg, I.Peák: Algebraic theory of automata, 1972
- V.N.Salij: Universal algebra and automata, in Russian, 1988
- J.M.Hovie: Automata and languages, 1991
- J.Hromkovič: Theoretical computer science. Introduction to automata, computability, complexity, algorithmics, randomization, communication, and cryptography, 2004

Introduction

- S.Ginsburg: An introduction to mathematical machine theory, 1962
- F.Gécseg, I.Peák: Algebraic theory of automata, 1972
- V.N.Salij: Universal algebra and automata, in Russian, 1988
- J.M.Hovie: Automata and languages, 1991
- J.Hromkovič: Theoretical computer science. Introduction to automata, computability, complexity, algorithmics, randomization, communication, and cryptography, 2004
- M.Berthé, M.Rigi: Combinatorics, automata, and number theory, 2010

Introduction

Introduction

monographs or surveys in which unary and monounary algebras were studied:

- J. Johnson and R. L. Seifert: A survey of multi-unary algebras, 1967

Introduction

monographs or surveys in which unary and monounary algebras were studied:

- J. Johnson and R. L. Seifert: A survey of multi-unary algebras, 1967
- B. Jónsson: Topics in universal algebra, 1972

Introduction

monographs or surveys in which unary and monounary algebras were studied:

- J. Johnson and R. L. Seifert: A survey of multi-unary algebras, 1967
- B. Jónsson: Topics in universal algebra, 1972
- L. A. Skornjakov: Unars, 1982

Introduction

monographs or surveys in which unary and monounary algebras were studied:

- J. Johnson and R. L. Seifert: A survey of multi-unary algebras, 1967
- B. Jónsson: Topics in universal algebra, 1972
- L. A. Skornjakov: Unars, 1982
- M. Novotný: Mono-unary algebras in the work of Czechoslovak mathematicians, 1990

Introduction

monographs or surveys in which unary and monounary algebras were studied:

- J. Johnson and R. L. Seifert: A survey of multi-unary algebras, 1967
- B. Jónsson: Topics in universal algebra, 1972
- L. A. Skornjakov: Unars, 1982
- M. Novotný: Mono-unary algebras in the work of Czechoslovak mathematicians, 1990
- J.Chvalina: Functional graphs, quasiordered sets and commutative hypergroups, 1995

Introduction

monographs or surveys in which unary and monounary algebras were studied:

- J. Johnson and R. L. Seifert: A survey of multi-unary algebras, 1967
- B. Jónsson: Topics in universal algebra, 1972
- L. A. Skornjakov: Unars, 1982
- M. Novotný: Mono-unary algebras in the work of Czechoslovak mathematicians, 1990
- J.Chvalina: Functional graphs, quasiordered sets and commutative hypergroups, 1995
- J. G. Pitkethly, B. A. Davey: Dualisability: Unary Algebras and Beyond, 2005

Introduction

monographs or surveys in which unary and monounary algebras were studied:

- J. Johnson and R. L. Seifert: A survey of multi-unary algebras, 1967
- B. Jónsson: Topics in universal algebra, 1972
- L. A. Skornjakov: Unars, 1982
- M. Novotný: Mono-unary algebras in the work of Czechoslovak mathematicians, 1990
- J.Chvalina: Functional graphs, quasiordered sets and commutative hypergroups, 1995
- J. G. Pitkethly, B. A. Davey: Dualisability: Unary Algebras and Beyond, 2005
- D. Jakubíková-Studenovská, J. Pócs: Monounary algebras, 2009

Introduction

Introduction

UNARY ALGEBRAS

Introduction

UNARY ALGEBRAS

subject **08A60**

- more than forty years

Introduction

UNARY ALGEBRAS

subject **08A60**

- more than forty years
- in the official AMS Classification

Introduction

UNARY ALGEBRAS

subject **08A60**

- more than forty years
- in the official AMS Classification

- 1969 J. S. Johnson *monounary*

Introduction

UNARY ALGEBRAS

subject **08A60**

- more than forty years
- in the official AMS Classification

- 1969 J. S. Johnson *monounary*
- 1960 J. G. Marica, S. J. Bryant *unary*

Automata

Automata

A **Mealy-type automaton** is a system $\mathcal{A} = (A, X, Y, \delta, \lambda)$, where A, X, Y are nonempty sets, $\delta : A \times X \to A$ and $\lambda : A \times X \to Y$ are functions defined on $A \times X$. Then A is called a set of **(internal) states**, X is called a set of **inputs** and Y a set of **outputs**. The function δ is said to be a **transition function** (or a **next state function**) and λ is an **output function**.

Automata

A **Mealy-type automaton** is a system $\mathcal{A} = (A, X, Y, \delta, \lambda)$, where A, X, Y are nonempty sets, $\delta : A \times X \to A$ and $\lambda : A \times X \to Y$ are functions defined on $A \times X$. Then A is called a set of **(internal) states**, X is called a set of **inputs** and Y a set of **outputs**. The function δ is said to be a **transition function** (or a **next state function**) and λ is an **output function**.

WORK:

(1) **A** is in a state $a \in A$

Automata

A **Mealy-type automaton** is a system $\mathcal{A} = (A, X, Y, \delta, \lambda)$, where A, X, Y are nonempty sets, $\delta : A \times X \to A$ and $\lambda : A \times X \to Y$ are functions defined on $A \times X$. Then A is called a set of **(internal) states**, X is called a set of **inputs** and Y a set of **outputs**. The function δ is said to be a **transition function** (or a **next state function**) and λ is an **output function**.

WORK:

(1) **A** is in a state $a \in A$

(2) input $x \in X$ is applied to a

Automata

A **Mealy-type automaton** is a system $\mathcal{A} = (A, X, Y, \delta, \lambda)$, where A, X, Y are nonempty sets, $\delta : A \times X \to A$ and $\lambda : A \times X \to Y$ are functions defined on $A \times X$. Then A is called a set of **(internal) states**, X is called a set of **inputs** and Y a set of **outputs**. The function δ is said to be a **transition function** (or a **next state function**) and λ is an **output function**.

WORK:

(1) **A** is in a state $a \in A$

(2) input $x \in X$ is applied to a

(3) **A** goes into a state $\delta(a, x) \in A$

Automata

A **Mealy-type automaton** is a system $\mathcal{A} = (A, X, Y, \delta, \lambda)$, where A, X, Y are nonempty sets, $\delta : A \times X \to A$ and $\lambda : A \times X \to Y$ are functions defined on $A \times X$. Then A is called a set of **(internal) states**, X is called a set of **inputs** and Y a set of **outputs**. The function δ is said to be a **transition function** (or a **next state function**) and λ is an **output function**.

WORK:

(1) **A** is in a state $a \in A$

(2) input $x \in X$ is applied to a

(3) **A** goes into a state $\delta(a, x) \in A$

(4) by transition, **A** sends out an output $\lambda(a, x) \in Y$

Automata

Automata

- **Moore-type automaton**: a Mealy-type automaton with

$$\delta(a, x) = \delta(a', x') \implies \lambda(a, x) = \lambda(a', x') \forall\, a, a' \in A, x, x' \in X$$

μ with $\lambda(a, x) = \mu(\delta(a, x))$ is a *sign function* of **A**

Automata

- **Moore-type automaton**: a Mealy-type automaton with

  $$\delta(a,x)=\delta(a',x') \implies \lambda(a,x)=\lambda(a',x')\forall\, a,a'\in A, x,x'\in X$$

  μ with $\lambda(a,x)=\mu(\delta(a,x))$ is a *sign function* of **A**
- **automaton without outputs**: a Moore-type automaton **A** such that $Y=A$ and $\delta(a,x)=\lambda(a,x)$ for each $a\in A$, $x\in X$; denote

  $$\mathbf{A}=(A,X,\delta)$$

Automata

- **Moore-type automaton**: a Mealy-type automaton with

  $$\delta(a,x) = \delta(a',x') \implies \lambda(a,x) = \lambda(a',x') \forall\, a, a' \in A, x, x' \in X$$

  μ with $\lambda(a,x) = \mu(\delta(a,x))$ is a *sign function* of **A**
- **automaton without outputs**: a Moore-type automaton **A** such that $Y = A$ and $\delta(a,x) = \lambda(a,x)$ for each $a \in A$, $x \in X$; denote

  $$\mathbf{A} = (A, X, \delta)$$
- **autonomous automaton**: a Moore-type automaton without outputs and such that X is a one-element set

Automata

- finite Mealy-type automaton $\mathbf{A} = (A, X, Y, \delta, \lambda)$

Automata

- finite Mealy-type automaton $\mathbf{A} = (A, X, Y, \delta, \lambda)$
- To present it - make a table: *transition-output table*

Automata

- finite Mealy-type automaton $\mathbf{A} = (A, X, Y, \delta, \lambda)$
- To present it - make a table: *transition-output table*
  - rows indicated by inputs

Automata

- finite Mealy-type automaton $\mathbf{A} = (A, X, Y, \delta, \lambda)$
- To present it - make a table: *transition-output table*
  - rows indicated by inputs
  - columns indicated by states

Automata

- finite Mealy-type automaton $\mathbf{A} = (A, X, Y, \delta, \lambda)$
- To present it - make a table: *transition-output table*
  - rows indicated by inputs
  - columns indicated by states
  - elements of the matrix inside are vectors $(\delta(a, x), \lambda(a, x))$

Automata

- finite Mealy-type automaton $\mathbf{A} = (A, X, Y, \delta, \lambda)$
- To present it - make a table: *transition-output table*
  - rows indicated by inputs
  - columns indicated by states
  - elements of the matrix inside are vectors $(\delta(a, x), \lambda(a, x))$
- finite Moore-type automaton

Automata

- finite Mealy-type automaton $\mathbf{A} = (A, X, Y, \delta, \lambda)$
- To present it - make a table: *transition-output table*
    - rows indicated by inputs
    - columns indicated by states
    - elements of the matrix inside are vectors $(\delta(a, x), \lambda(a, x))$
- finite Moore-type automaton
    - rows indicated by inputs

Automata

- finite Mealy-type automaton $\mathbf{A} = (A, X, Y, \delta, \lambda)$
- To present it - make a table: *transition-output table*
  - rows indicated by inputs
  - columns indicated by states
  - elements of the matrix inside are vectors $(\delta(a, x), \lambda(a, x))$
- finite Moore-type automaton
  - rows indicated by inputs
  - columns indicated by states

Automata

- finite Mealy-type automaton $\mathbf{A} = (A, X, Y, \delta, \lambda)$
- To present it - make a table: *transition-output table*
  - rows indicated by inputs
  - columns indicated by states
  - elements of the matrix inside are vectors $(\delta(a, x), \lambda(a, x))$
- finite Moore-type automaton
  - rows indicated by inputs
  - columns indicated by states
  - elements of the matrix inside are $\delta(a, x)$

Automata

- finite Mealy-type automaton $\mathbf{A} = (A, X, Y, \delta, \lambda)$
- To present it - make a table: *transition-output table*
  - rows indicated by inputs
  - columns indicated by states
  - elements of the matrix inside are vectors $(\delta(a, x), \lambda(a, x))$
- finite Moore-type automaton
  - rows indicated by inputs
  - columns indicated by states
  - elements of the matrix inside are $\delta(a, x)$
  - above the column indicated by $a \in A$, the sign $\mu(a)$

Automata

- finite Mealy-type automaton $\mathbf{A} = (A, X, Y, \delta, \lambda)$
- To present it - make a table: *transition-output table*
  - rows indicated by inputs
  - columns indicated by states
  - elements of the matrix inside are vectors $(\delta(a, x), \lambda(a, x))$
- finite Moore-type automaton
  - rows indicated by inputs
  - columns indicated by states
  - elements of the matrix inside are $\delta(a, x)$
  - above the column indicated by $a \in A$, the sign $\mu(a)$
- finite Moore-type automaton without outputs: transition-output table is reduced to a *transition table*

Automata

- F.Gécseg, I.Peák:

"The difference between the working of Mealy- and Moore-type automata can be interpreted as follows: the output of a Mealy-type automaton occurs **while** the automaton is going to the next state, and the output of a Moore-type automaton occurs **after** the automaton has gone into a next state and the output is the sign of this new state. On the other hand, an automaton without outputs behaves as a closed box: for an input sign it reacts by changing its internal state only. As will be seen later, **from the point of view of the algebraic theory**, the concept of the Moore-type automaton is merely an apparent specialization of that of the Mealy-type automaton. Furthermore, **it should be mentioned that in many cases we may confine ourselves to automata without outputs.**"

Monounary algebras

Monounary algebras

monounary algebra (A, f): A - nonempty set, f - mapping of A into A

- **connected**: $\forall\ x, y \in A\ \exists\ n, m \in \mathbb{N} \cup \{0\}$ such that

$$f^n(x) = f^m(y)$$

Monounary algebras

monounary algebra (A, f): A - nonempty set, f - mapping of A into A

- **connected**: $\forall\ x, y \in A\ \exists\ n, m \in \mathbb{N} \cup \{0\}$ such that
$$f^n(x) = f^m(y)$$
- **connected component** of (A, f): maximal connected subalgebra

Monounary algebras

monounary algebra (A, f): A - nonempty set, f - mapping of A into A

- **connected**: $\forall\ x, y \in A\ \exists\ n, m \in \mathbb{N} \cup \{0\}$ such that

$$f^n(x) = f^m(y)$$

- **connected component** of (A, f): maximal connected subalgebra
- $c \in A$ is **cyclic** if $f^k(c) = c$ for some $k \in N$

Monounary algebras

monounary algebra (A, f): A - nonempty set, f - mapping of A into A

- **connected**: $\forall\ x, y \in A\ \exists\ n, m \in \mathbb{N} \cup \{0\}$ such that
$$f^n(x) = f^m(y)$$
- **connected component** of (A, f): maximal connected subalgebra
- $c \in A$ is **cyclic** if $f^k(c) = c$ for some $k \in N$
- the set of all cyclic elements of some connected component of (A, f) is a **cycle** of (A, f)

Monounary algebras

Monounary algebras

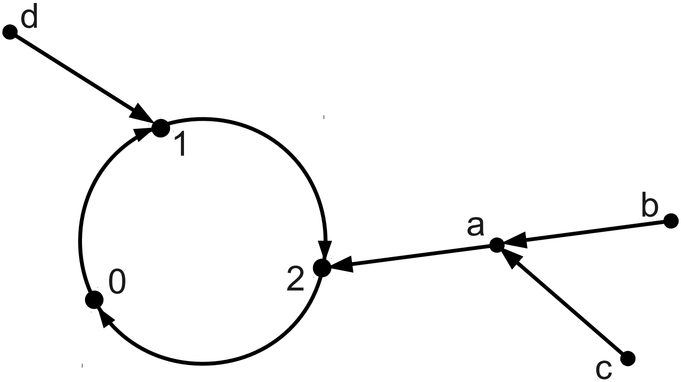


Figure:

Automata without outputs and unary algebras

Automata without outputs and unary algebras

- $\mathbf{A} = (A, X, \delta)$ be a **Moore-type automaton without outputs**

Automata without outputs and unary algebras

- $\mathbf{A} = (A, X, \delta)$ be a **Moore-type automaton without outputs**
  - for $x \in X$ define a unary operation f_x:

$$f_x(a) = \delta(a, x) \text{ for each } a \in A$$
$$F = \{f_x : x \in X\}$$

Automata without outputs and unary algebras

- $\mathbf{A} = (A, X, \delta)$ be a **Moore-type automaton without outputs**
  - for $x \in X$ define a unary operation f_x:

$$f_x(a) = \delta(a, x) \text{ for each } a \in A$$
$$F = \{f_x : x \in X\}$$

  - (A, F) is a **unary algebra**; denote it $u(\mathbf{A})$

Automata without outputs and unary algebras

- $\mathbf{A} = (A, X, \delta)$ be a **Moore-type automaton without outputs**
  - for $x \in X$ define a unary operation f_x:

$$f_x(a) = \delta(a, x) \text{ for each } a \in A$$
$$F = \{f_x : x \in X\}$$

  - (A, F) is a **unary algebra**; denote it $u(\mathbf{A})$

- $\mathcal{A} = (A, \{g_j : j \in J\})$ be a **unary algebra**

Automata without outputs and unary algebras

- $\mathbf{A} = (A, X, \delta)$ be a **Moore-type automaton without outputs**
    - for $x \in X$ define a unary operation f_x:

$$f_x(a) = \delta(a, x) \text{ for each } a \in A$$
$$F = \{f_x : x \in X\}$$

    - (A, F) is a **unary algebra**; denote it $u(\mathbf{A})$

- $\mathcal{A} = (A, \{g_j : j \in J\})$ be a **unary algebra**
    - for $a \in A$, $j \in J$ define

$$\delta(a, j) = g_j(a)$$

Automata without outputs and unary algebras

- $\mathbf{A} = (A, X, \delta)$ be a **Moore-type automaton without outputs**
  - for $x \in X$ define a unary operation f_x:

$$f_x(a) = \delta(a, x) \text{ for each } a \in A$$
$$F = \{f_x : x \in X\}$$

  - (A, F) is a **unary algebra**; denote it $u(\mathbf{A})$

- $\mathcal{A} = (A, \{g_j : j \in J\})$ be a **unary algebra**
  - for $a \in A$, $j \in J$ define

$$\delta(a, j) = g_j(a)$$

  - (A, J, δ) is a **Moore-type automaton without outputs**; denote it $M(\mathcal{A})$

Automata without outputs and unary algebras

Automata without outputs and unary algebras

- correspondence between the class of all Moore-type automata without outputs and the class of all unary algebras

Automata without outputs and unary algebras

- correspondence between the class of all Moore-type automata without outputs and the class of all unary algebras

Theorem

(i) *If* $\mathbf{A}$ *is a Moore-type automaton without outputs, then* $u(\mathbf{A})$ *is a unary algebra and* $M(u(\mathbf{A})) \cong \mathbf{A}$*;*

Automata without outputs and unary algebras

- correspondence between the class of all Moore-type automata without outputs and the class of all unary algebras

Theorem

(i) *If* $\mathbf{A}$ *is a Moore-type automaton without outputs, then* $u(\mathbf{A})$ *is a unary algebra and* $M(u(\mathbf{A})) \cong \mathbf{A}$*;*

(ii) *if* $\mathcal{A}$ *is a unary algebra, then* $M(\mathcal{A})$ *is a Moore-type automaton without outputs and* $u(M(\mathcal{A})) \cong \mathcal{A}$*.*

Automata without outputs and unary algebras

- correspondence between the class of all Moore-type automata without outputs and the class of all unary algebras

Theorem

(i) *If* $\mathbf{A}$ *is a Moore-type automaton without outputs, then* $u(\mathbf{A})$ *is a unary algebra and* $M(u(\mathbf{A})) \cong \mathbf{A}$*;*

(ii) *if* $\mathcal{A}$ *is a unary algebra, then* $M(\mathcal{A})$ *is a Moore-type automaton without outputs and* $u(M(\mathcal{A})) \cong \mathcal{A}$*.*

- correspondence between the class of all autonomous automata and the class of all monounary algebras

Automata without outputs and unary algebras

- correspondence between the class of all Moore-type automata without outputs and the class of all unary algebras

Theorem

(i) *If* $\mathbf{A}$ *is a Moore-type automaton without outputs, then* $u(\mathbf{A})$ *is a unary algebra and* $M(u(\mathbf{A})) \cong \mathbf{A}$*;*

(ii) *if* $\mathcal{A}$ *is a unary algebra, then* $M(\mathcal{A})$ *is a Moore-type automaton without outputs and* $u(M(\mathcal{A})) \cong \mathcal{A}$*.*

- correspondence between the class of all autonomous automata and the class of all monounary algebras

Theorem

(i') *If* $\mathbf{A}$ *is an autonomous automaton, then* $u(\mathbf{A})$ *is a monounary algebra and* $M(u(\mathbf{A})) \cong \mathbf{A}$*;*

Automata without outputs and unary algebras

- correspondence between the class of all Moore-type automata without outputs and the class of all unary algebras

Theorem

(i) *If* $\mathbf{A}$ *is a Moore-type automaton without outputs, then* $u(\mathbf{A})$ *is a unary algebra and* $M(u(\mathbf{A})) \cong \mathbf{A}$;

(ii) *if* $\mathcal{A}$ *is a unary algebra, then* $M(\mathcal{A})$ *is a Moore-type automaton without outputs and* $u(M(\mathcal{A})) \cong \mathcal{A}$.

- correspondence between the class of all autonomous automata and the class of all monounary algebras

Theorem

(i') *If* $\mathbf{A}$ *is an autonomous automaton, then* $u(\mathbf{A})$ *is a monounary algebra and* $M(u(\mathbf{A})) \cong \mathbf{A}$;

(ii') *if* $\mathcal{A}$ *is a monounary algebra, then* $M(\mathcal{A})$ *is an autonomous automaton and* $u(M(\mathcal{A})) \cong \mathcal{A}$.

Monounary algebras imply other algebras: Cardinality

Monounary algebras imply other algebras: Cardinality

- Investigate questions, in which some assertion or construction is shown for **algebras of several types**, and by the proof there is applied a **sooner** shown assertion for **monounary algebras**

Monounary algebras imply other algebras: Cardinality

- Investigate questions, in which some assertion or construction is shown for **algebras of several types**, and by the proof there is applied a **sooner** shown assertion for **monounary algebras**
- How many algebras of type F there exist on the set A, i.e., how many possibilities we have to choose values of operations on A, if

Monounary algebras imply other algebras: Cardinality

- Investigate questions, in which some assertion or construction is shown for **algebras of several types**, and by the proof there is applied a **sooner** shown assertion for **monounary algebras**
- How many algebras of type F there exist on the set A, i.e., how many possibilities we have to choose values of operations on A, if
  - A - set of cardinality m
  - $F = \{f_j : j \in J\}$ - a set of operation symbols
  - $ar(f) \in \mathbb{N} \cup \{0\}$ - arity of $f \in F$
  - not all $f \in F$ are nullary

Monounary algebras imply other algebras: Cardinality

Monounary algebras imply other algebras: Cardinality

- For $k \in \mathbb{N} \cup \{0\}$ let n_k be the number of k-ary operation symbols of F.

Monounary algebras imply other algebras: Cardinality

- For $k \in \mathbb{N} \cup \{0\}$ let n_k be the number of k-ary operation symbols of F.

Lemma

The number of operations of arity k over A is equal to m^{m^k}.

Monounary algebras imply other algebras: Cardinality

- For $k \in \mathbb{N} \cup \{0\}$ let n_k be the number of k-ary operation symbols of F.

Lemma

The number of operations of arity k over A is equal to m^{m^k}.

Lemma

There exist $m^{\sum_{k \in \mathbb{N} \cup \{0\}} n_k m^k}$ of algebras over A of type F.

Monounary algebras imply other algebras: Cardinality

- For $k \in \mathbb{N} \cup \{0\}$ let n_k be the number of k-ary operation symbols of F.

Lemma

The number of operations of arity k over A is equal to m^{m^k}.

Lemma

There exist $m^{\sum_{k \in \mathbb{N} \cup \{0\}} n_k m^k}$ *of algebras over A of type F.*

Lemma

If m is infinite and $p = \sum_{k \in \mathbb{N}} n_k$, *then there exist* $m^{n_0}.2^{mp}$ *of algebras over A of type F.*

Monounary algebras imply other algebras: Cardinality

- For $k \in \mathbb{N} \cup \{0\}$ let n_k be the number of k-ary operation symbols of F.

Lemma

The number of operations of arity k over A is equal to m^{m^k}.

Lemma

There exist $m^{\sum_{k \in \mathbb{N} \cup \{0\}} n_k m^k}$ of algebras over A of type F.

Lemma

If m is infinite and $p = \sum_{k \in \mathbb{N}} n_k$, then there exist $m^{n_0}.2^{mp}$ of algebras over A of type F.

- Thus this is an upper bound for the number of **non-isomorphic** algebras of the form (A, F).

Monounary algebras imply other algebras: Cardinality

Monounary algebras imply other algebras: Cardinality

- How many non-isomorphic monounary algebras on an n-element set?

Monounary algebras imply other algebras: Cardinality

- How many non-isomorphic monounary algebras on an n-element set?
  - $$n = 1 \qquad \Longrightarrow \qquad 1$$

Monounary algebras imply other algebras: Cardinality

- How many non-isomorphic monounary algebras on an n-element set?
  - $$n = 1 \quad \Longrightarrow \quad 1$$
  - $$n = 2 \quad \Longrightarrow \quad 3$$

Monounary algebras imply other algebras: Cardinality

- How many non-isomorphic monounary algebras on an n-element set?
  - $$n = 1 \implies 1$$
  - $$n = 2 \implies 3$$
  - $$n = 3 \implies 7$$

Monounary algebras imply other algebras: Cardinality

- How many non-isomorphic monounary algebras on an n-element set?
  - $$n = 1 \implies 1$$
  - $$n = 2 \implies 3$$
  - $$n = 3 \implies 7$$
  - $$n = 4 \implies 18$$

Monounary algebras imply other algebras: Cardinality

- How many non-isomorphic monounary algebras on an n-element set?
  - $$n = 1 \implies 1$$
  - $$n = 2 \implies 3$$
  - $$n = 3 \implies 7$$
  - $$n = 4 \implies 18$$
  - $$n = 5 \implies 37$$

Monounary algebras imply other algebras: Cardinality

Monounary algebras imply other algebras: Cardinality

S. D. Comer, J. J. LeTourneau, 1969:

Theorem

For every infinite cardinal m there exist 2^m pairwise non-isomorphic m-element algebras with trivial automorphism groups.

Monounary algebras imply other algebras: Cardinality

S. D. Comer, J. J. LeTourneau, 1969:

Theorem

For every infinite cardinal m there exist 2^m pairwise non-isomorphic m-element algebras with trivial automorphism groups.

Theorem

If m is infinite, then on A there exist $m^{n_0}.2^{mp}$ pairwise non-isomorphic algebras with trivial automorphism groups.

Proof.

$\exists\ j_0 \in J$ with $k = ar(f_{j_0}) > 0$. By the above theorem $\exists\ 2^m$ pairwise non-isomorphic algebras (A, g_i), $i \in I$ with trivial automorphism groups (g_i of arity k). Take all algebras (A, F) such that $f_{j_0} = g_i$ for some $i \in I$. Each f_j can be chosen in m ways if $ar(f_j) = 0$, and in 2^m ways if $ar(f_j) > 0$. The number of distinct obtained structures is $m^{n_0}.2^{mp}$. They have trivial automorphism groups, they are non-isomorphic. $\square$

Monounary algebras

Monounary algebras

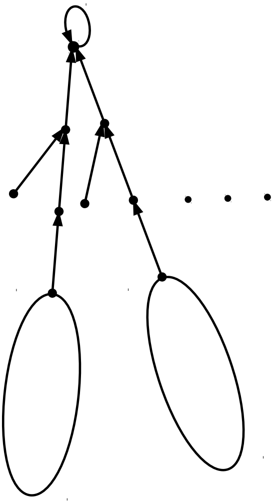


Figure:

Monounary algebras imply other algebras: Homomorphisms

Monounary algebras imply other algebras: Homomorphisms

- O. Borůvka: How to find all matrices commuting with a given matrix?
- How to find all linear transformations of an n-dimensional vector space that commute with a given linear transformation?

Monounary algebras imply other algebras: Homomorphisms

- O. Borůvka: How to find all matrices commuting with a given matrix?
- How to find all linear transformations of an n-dimensional vector space that commute with a given linear transformation?
- Neglecting the linear structure of the vector space and of linear transformations: make a **construction providing exactly all homomorphisms of one monounary algebra into another one**.

Monounary algebras imply other algebras: Homomorphisms

Monounary algebras imply other algebras: Homomorphisms

- Classical result of M. Novotný which resolves this problem:
  - Introduced the notion of a monounary algebra which is *admissible* to a given connected monounary algebra. Further, if (A, f) and (B, g) are monounary algebras, conditions under which there exists a homomorphism of (A, f) to (B, g) were found and all homomorphisms (A, f) to (B, g) were constructively described.

Monounary algebras imply other algebras: Homomorphisms

- Classical result of M. Novotný which resolves this problem:
  - Introduced the notion of a monounary algebra which is *admissible* to a given connected monounary algebra. Further, if (A, f) and (B, g) are monounary algebras, conditions under which there exists a homomorphism of (A, f) to (B, g) were found and all homomorphisms (A, f) to (B, g) were constructively described.
  - For connected monounary algebras, a homomorphism of (A, f) to (B, g) exists if and only if (B, g) is admissible to (A, f).

Monounary algebras imply other algebras: Homomorphisms

Monounary algebras imply other algebras: Homomorphisms

- The acquaintance of the mentioned construction for monounary algebras can be used for arbitrary algebraic structures. To a given algebraic structure, a monounary algebra is assigned. Then instead of looking for **homomorphisms between algebraic structures** we can look for **homomorphisms of the corresponding algebras**.

Monounary algebras imply other algebras: Homomorphisms

- The acquaintance of the mentioned construction for monounary algebras can be used for arbitrary algebraic structures. To a given algebraic structure, a monounary algebra is assigned. Then instead of looking for **homomorphisms between algebraic structures** we can look for **homomorphisms of the corresponding algebras**.
- Illustration on groupoids:
  - given $\mathcal{G} = (G, \circ)$
  - **assigned** monounary algebra $un(\mathcal{G}) = (G \times G, f)$, if $f((x, y)) = (y, x \circ y)$ for any $(x, y) \in G \times G$

Monounary algebras imply other algebras: Homomorphisms

- The acquaintance of the mentioned construction for monounary algebras can be used for arbitrary algebraic structures. To a given algebraic structure, a monounary algebra is assigned. Then instead of looking for **homomorphisms between algebraic structures** we can look for **homomorphisms of the corresponding algebras**.
- Illustration on groupoids:
  - given $\mathcal{G} = (G, \circ)$
  - **assigned** monounary algebra $un(\mathcal{G}) = (G \times G, f)$, if $f((x, y)) = (y, x \circ y)$ for any $(x, y) \in G \times G$
- G, H - sets, $\varphi : G \to H$, define $\varphi \times \varphi = \psi : G \times G \to H \times H$ with $\psi((x, y)) = (\varphi(x), \varphi(y))\ \forall (x, y) \in G \times G$
- $\psi : G \times G \to H \times H$ will be called **decomposable** if there is $\varphi : G \to H$ such that $\psi = \varphi \times \varphi$

Monounary algebras imply other algebras: Homomorphisms

Monounary algebras imply other algebras: Homomorphisms

Construction for finding all homomorphisms between groupoids $(G, \circ)$, $(H, *)$.

Monounary algebras imply other algebras: Homomorphisms

Construction for finding all homomorphisms between groupoids $(G, \circ)$, $(H, *)$.

(1) Construct $un(G, \circ) = (G \times G, f)$, $un(H, *) = (H \times H, g)$.

Monounary algebras imply other algebras: Homomorphisms

Construction for finding all homomorphisms between groupoids $(G, \circ)$, $(H, *)$.

(1) Construct $un(G, \circ) = (G \times G, f)$, $un(H, *) = (H \times H, g)$.

(2) Construct all homomorphisms of $(G \times G, f)$ into $(H \times H, g)$.

Monounary algebras imply other algebras: Homomorphisms

Construction for finding all homomorphisms between groupoids $(G, \circ)$, $(H, *)$.

(1) Construct $un(G, \circ) = (G \times G, f)$, $un(H, *) = (H \times H, g)$.

(2) Construct all homomorphisms of $(G \times G, f)$ into $(H \times H, g)$.

(3) Test all the constructed homomorphisms and reject all that are not decomposable.

Monounary algebras imply other algebras: Homomorphisms

Construction for finding all homomorphisms between groupoids $(G, \circ)$, $(H, *)$.

(1) Construct $un(G, \circ) = (G \times G, f)$, $un(H, *) = (H \times H, g)$.

(2) Construct all homomorphisms of $(G \times G, f)$ into $(H \times H, g)$.

(3) Test all the constructed homomorphisms and reject all that are not decomposable.

(4) For any decomposable homomorphism ψ of $(G \times G, f)$ into $(H \times H, g)$ construct φ such that $\psi = \varphi \times \varphi$.

Monounary algebras imply other algebras: Homomorphisms

Construction for finding all homomorphisms between groupoids $(G, \circ)$, $(H, *)$.

(1) Construct $un(G, \circ) = (G \times G, f)$, $un(H, *) = (H \times H, g)$.

(2) Construct all homomorphisms of $(G \times G, f)$ into $(H \times H, g)$.

(3) Test all the constructed homomorphisms and reject all that are not decomposable.

(4) For any decomposable homomorphism ψ of $(G \times G, f)$ into $(H \times H, g)$ construct φ such that $\psi = \varphi \times \varphi$.

Then φ is a homomorphism of $(G, \circ))$ into $(H, *)$ and any homomorphism of $(G, \circ))$ into $(H, *)$ can be constructed in this way.

Monounary algebras imply other algebras: Retracts

Monounary algebras imply other algebras: Retracts

- retracts investigated in many areas of mathematics: topological spaces, later algebraic structures as groups, lattices, posets, etc.
- A substructure $\mathcal{B}$ of a structure $\mathcal{A}$ is said to be a *retract* of $\mathcal{A}$ if there exists an endomorphism (called a *retraction*) ϕ of $\mathcal{A}$ onto $\mathcal{B}$ such that $\phi(b) = b$ for each element b of $\mathcal{B}$.

Monounary algebras imply other algebras: Retracts

- retracts investigated in many areas of mathematics: topological spaces, later algebraic structures as groups, lattices, posets, etc.
- A substructure $\mathcal{B}$ of a structure $\mathcal{A}$ is said to be a *retract* of $\mathcal{A}$ if there exists an endomorphism (called a *retraction*) ϕ of $\mathcal{A}$ onto $\mathcal{B}$ such that $\phi(b) = b$ for each element b of $\mathcal{B}$.
- M.Novotný 2007: A construction of all retracts of a general algebra can be reduced to a construction of all retracts of a special monounary algebra.

Monounary algebras contra (some) other algebras: Abstract representation of automorphism groups

Monounary algebras contra (some) other algebras: Abstract representation of automorphism groups

$\mathcal{A}$ - algebra, system $End(\mathcal{A})$ of all endomorphisms of $\mathcal{A}$ is a monoid and the system $Aut(\mathcal{A})$ of all automorphisms of $\mathcal{A}$ is a group

Monounary algebras contra (some) other algebras: Abstract representation of automorphism groups

$\mathcal{A}$ - algebra, system $End(\mathcal{A})$ of all endomorphisms of $\mathcal{A}$ is a monoid and the system $Aut(\mathcal{A})$ of all automorphisms of $\mathcal{A}$ is a group

Problems of abstract representation:

(G) Is every group isomorphic to the group $Aut(\mathcal{A})$ for some algebraic structure $\mathcal{A}$?

(M) Is every monoid isomorphic to the monoid $End(\mathcal{A})$ for some algebraic structure $\mathcal{A}$?

Abstract representation of automorphism groups

Abstract representation of automorphism groups

Theorem

Each monoid is isomorphic to End$(\mathcal{A})$ for some unary algebra $\mathcal{A}$.

Abstract representation of automorphism groups

Theorem

Each monoid is isomorphic to $End(\mathcal{A})$ *for some unary algebra* $\mathcal{A}$.

Corollary

Each group is isomorphic to $Aut(\mathcal{A})$ *for some unary algebra* $\mathcal{A}$.

Abstract representation of automorphism groups

Theorem

Each monoid is isomorphic to End($\mathcal{A}$) for some unary algebra $\mathcal{A}$.

Corollary

Each group is isomorphic to Aut($\mathcal{A}$) for some unary algebra $\mathcal{A}$.

Abstract representation of automorphism groups

Theorem

Each monoid is isomorphic to $End(\mathcal{A})$ for some unary algebra $\mathcal{A}$.

Proof.

$(M, \cdot, e)$ be a monoid. Denote by $\mathcal{A} = (M, F)$, $F = \{f_m : m \in M\}$, where $f_m(x) = x \cdot m$ for each $x \in M$. $\square$

Corollary

Each group is isomorphic to $Aut(\mathcal{A})$ for some unary algebra $\mathcal{A}$.

Abstract representation of automorphism groups

Abstract representation of automorphism groups

G. Birkhoff 1946: to each group there exists

- a partially ordered set $\mathcal{A}$ with $G \cong Aut(\mathcal{A})$,

Abstract representation of automorphism groups

G. Birkhoff 1946: to each group there exists

- a partially ordered set $\mathcal{A}$ with $G \cong Aut(\mathcal{A})$,
- a distributive lattice $\mathcal{A}$ with $G \cong Aut(\mathcal{A})$,

Abstract representation of automorphism groups

G. Birkhoff 1946: to each group there exists

- a partially ordered set $\mathcal{A}$ with $G \cong Aut(\mathcal{A})$,
- a distributive lattice $\mathcal{A}$ with $G \cong Aut(\mathcal{A})$,
- a semilattice $\mathcal{A}$ with $G \cong Aut(\mathcal{A})$.

Abstract representation of automorphism groups

Abstract representation of automorphism groups

P. Vopěnka, A. Pultr, Z. Hedrlín, J. Sichler: for any monoid M and any cardinal n there exists

- a graph of order at least n whose endomorphism monoid is isomorphic to M
- a biunary algebra …
- a commutative groupoid …
- a symmetric graph …
- a semigroup …

Abstract representation of automorphism groups

P. Vopěnka, A. Pultr, Z. Hedrlín, J. Sichler: for any monoid M and any cardinal n there exists

- a graph of order at least n whose endomorphism monoid is isomorphic to M
- a biunary algebra ...
- a commutative groupoid ...
- a symmetric graph ...
- a semigroup ...

Theorem

(G.Fuhrken) There exists a group that is not isomorphic to the automorphism group of any monounary algebra.

Monounary algebras contra (some) other algebras: Retract Theorem

Monounary algebras contra (some) other algebras: Retract Theorem

- An element t of a structure $\mathcal{A}$ is said to be a *test element* if for any endomorphism ϕ of $\mathcal{A}$, $\phi(t) = t$ implies that ϕ is an automorphism.

Monounary algebras contra (some) other algebras: Retract Theorem

- An element t of a structure $\mathcal{A}$ is said to be a *test element* if for any endomorphism ϕ of $\mathcal{A}$, $\phi(t) = t$ implies that ϕ is an automorphism.
- The notion *test words* was first considered in the context of free groups: the first example Nielsen 1918: the commutator $[a, b] = aba^{-1}b^{-1}$ is a test word in the free group $F(a, b)$.

Monounary algebras contra (some) other algebras: Retract Theorem

- An element t of a structure $\mathcal{A}$ is said to be a *test element* if for any endomorphism ϕ of $\mathcal{A}$, $\phi(t) = t$ implies that ϕ is an automorphism.
- The notion *test words* was first considered in the context of free groups: the first example Nielsen 1918: the commutator $[a, b] = aba^{-1}b^{-1}$ is a test word in the free group $F(a, b)$.
- Test elements can be applied for **distinguishing automorphisms from non-automorphisms**: t is a test element if $\phi(t) = \alpha(t)$ for some automorphism α implies that ϕ is an automorphism. Thus the issue of deciding whether ϕ is an automorphism is replaced by that of deciding whether $\phi(t)$ and t are equivalent under the action of the automorphism group of a given structure.

Retract Theorem

Retract Theorem

- If t is a test element of $\mathcal{A}$, then t does not belong to any proper retract of $\mathcal{A}$.

Retract Theorem

- If t is a test element of $\mathcal{A}$, then t does not belong to any proper retract of $\mathcal{A}$.

E. C. Turner 1996 proved *Retract Theorem* for free groups:

Theorem

A word w in a free group F is a test word if and only if w is not in any proper retract of F.

Retract Theorem

- If t is a test element of $\mathcal{A}$, then t does not belong to any proper retract of $\mathcal{A}$.

E. C. Turner 1996 proved *Retract Theorem* for free groups:

Theorem

A word w in a free group F is a test word if and only if w is not in any proper retract of F.

- D. A. Voce 1995: there exists a group for which Retract Theorem fails to be valid (e.g. a fundamental group of the Klein bottle).

Retract Theorem

Retract Theorem

DJS, J. Pócs 2007:

- Retract Theorem is valid for each finite structure.

Retract Theorem

DJS, J. Pócs 2007:

- Retract Theorem is valid for each finite structure.
- There exists a unary algebra $\mathcal{A}$ with two operations such that $\mathcal{A}$ contains an element t which is not a test element and which does not belong to any proper retract of $\mathcal{A}$. Therefore for unary algebras with at least two operations Retract Theorem does not hold in general.

Retract Theorem

DJS, J. Pócs 2007:

- Retract Theorem is valid for each finite structure.
- There exists a unary algebra $\mathcal{A}$ with two operations such that $\mathcal{A}$ contains an element t which is not a test element and which does not belong to any proper retract of $\mathcal{A}$. Therefore for unary algebras with at least two operations Retract Theorem does not hold in general.
- Retract Theorem is valid for each monounary algebra.

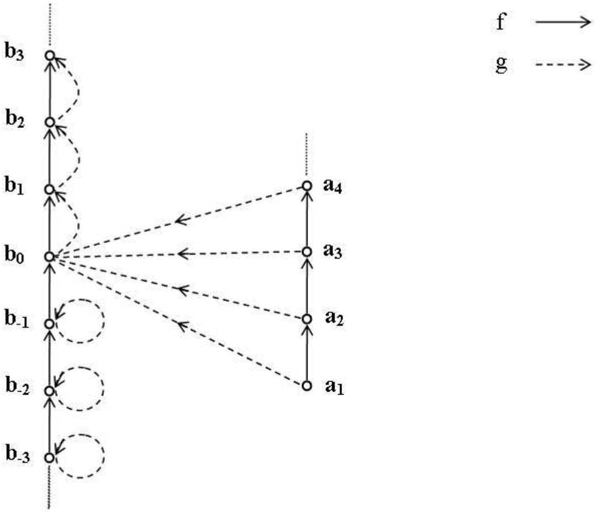


Figure:

Monounary algebras contra (some) other algebras: Convexities

Monounary algebras contra (some) other algebras: Convexities

- The notion of *convexity of lattices* is due to E. Fried; at the Problem Session of the Conference on General Algebra (Krems, 1988) he defined it as a nonempty class of lattices which is closed with respect to homomorphisms, direct products and convex sublattices.

Monounary algebras contra (some) other algebras: Convexities

- The notion of *convexity of lattices* is due to E. Fried; at the Problem Session of the Conference on General Algebra (Krems, 1988) he defined it as a nonempty class of lattices which is closed with respect to homomorphisms, direct products and convex sublattices.
- number of convexities of lattices?

Monounary algebras contra (some) other algebras: Convexities

- The notion of *convexity of lattices* is due to E. Fried; at the Problem Session of the Conference on General Algebra (Krems, 1988) he defined it as a nonempty class of lattices which is closed with respect to homomorphisms, direct products and convex sublattices.
- number of convexities of lattices?
- operators on a class $\mathcal{K}$ of algebras
  - $\mathbf{H}$ = forming homomorphic images
  - $\mathbf{S}$ = forming subalgebras
  - $\mathbf{P}$ = forming direct products

Convexities

Convexities

J. Jakubík 1992:

Theorem

The system of all convexities of lattices is a proper class.

Convexities

J. Jakubík 1992:

Theorem

The system of all convexities of lattices is a proper class.

- Convexities defined also for other types of algebraic structures, e.g., partially ordered groups as l-groups, d-groups, Riesz groups

Convexities

Convexities

- A class of partial monounary algebras is said to be a *convexity* if it is *closed under homomorphic images, direct products and convex relative subalgebras*

Convexities

- A class of partial monounary algebras is said to be a *convexity* if it is *closed under homomorphic images, direct products and convex relative subalgebras* (operators $\mathbf{H}, \mathbf{C}, \mathbf{P}$).

Convexities

- A class of partial monounary algebras is said to be a *convexity* if it is *closed under homomorphic images, direct products and convex relative subalgebras* (operators $\mathbf{H}, \mathbf{C}, \mathbf{P}$).

DJS 2009

Theorem

Let $\mathcal{T}$ be a class of partial monounary algebras. Then $\mathbf{HCP}\ \mathcal{T}$ *is the least convexity containing $\mathcal{T}$ (it is called a convexity generated by $\mathcal{T}$).*

Convexities

Convexities

K the collection of all convexities of partial monounary algebras (including the empty system); this collection is partially ordered by inclusion

Convexities

K the collection of all convexities of partial monounary algebras (including the empty system); this collection is partially ordered by inclusion

Theorem

The lattice **K** *consists of* 23 *convexities of partial monounary algebras;*

Convexities

K the collection of all convexities of partial monounary algebras (including the empty system); this collection is partially ordered by inclusion

Theorem

The lattice **K** *consists of* 23 *convexities of partial monounary algebras; the smallest one is* $\emptyset$, *the greatest (class of all partial monounary algebras) is generated by* $\mathbb{Z}$ *with the operation of successor.*

Convexities

K the collection of all convexities of partial monounary algebras (including the empty system); this collection is partially ordered by inclusion

Theorem

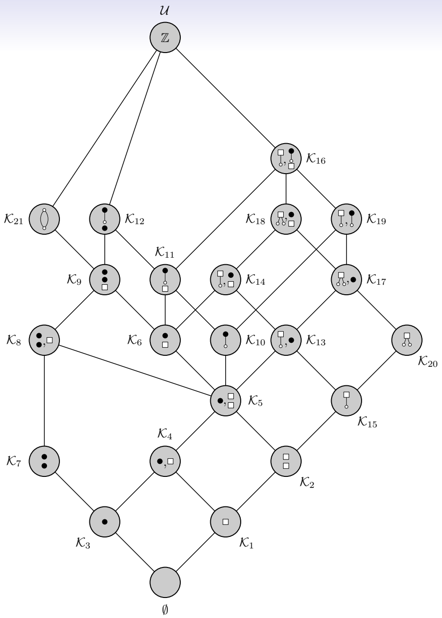
The lattice **K** *consists of* 23 *convexities of partial monounary algebras; the smallest one is* $\emptyset$*, the greatest (class of all partial monounary algebras) is generated by* $\mathbb{Z}$ *with the operation of successor. There are* 13 *principal convexities.*

Convexities

K the collection of all convexities of partial monounary algebras (including the empty system); this collection is partially ordered by inclusion

Theorem

The lattice **K** *consists of* 23 *convexities of partial monounary algebras; the smallest one is* $\emptyset$, *the greatest (class of all partial monounary algebras) is generated by* $\mathbb{Z}$ *with the operation of successor. There are* 13 *principal convexities. For each* $\mathcal{T} \in \mathbf{K}$ *there is* $\mathcal{T}_0$ *with at most two elements such that* $\mathcal{T}$ *is generated by* $\mathcal{T}_0$.



$\mathcal{U}$
$\mathbb{Z}$
$\mathcal{K}_{16}$
$\mathcal{K}_{21}$
$\mathcal{K}_{12}$
$\mathcal{K}_{18}$
$\mathcal{K}_{19}$
$\mathcal{K}_{11}$
$\mathcal{K}_{9}$
$\mathcal{K}_{14}$
$\mathcal{K}_{17}$
$\mathcal{K}_{8}$
$\mathcal{K}_{6}$
$\mathcal{K}_{10}$
$\mathcal{K}_{13}$
$\mathcal{K}_{20}$
$\mathcal{K}_{5}$
$\mathcal{K}_{15}$
$\mathcal{K}_{4}$
$\mathcal{K}_{7}$
$\mathcal{K}_{2}$
$\mathcal{K}_{3}$
$\mathcal{K}_{1}$
$\emptyset$

Figure:

Monounary algebras contra (some) other algebras: Pseudovarieties

Monounary algebras contra (some) other algebras: Pseudovarieties

- **variety** $=$ a class of those algebras which satisfy a given set of identities

Monounary algebras contra (some) other algebras: Pseudovarieties

- **variety** $=$ a class of those algebras which satisfy a given set of identities
- operators on a class $\mathcal{K}$ of algebras

Monounary algebras contra (some) other algebras: Pseudovarieties

- **variety** $=$ a class of those algebras which satisfy a given set of identities
- operators on a class $\mathcal{K}$ of algebras
  - **H**

Monounary algebras contra (some) other algebras: Pseudovarieties

- **variety** $=$ a class of those algebras which satisfy a given set of identities
- operators on a class $\mathcal{K}$ of algebras
  - **H**
  - **S**

Monounary algebras contra (some) other algebras: Pseudovarieties

- **variety** $=$ a class of those algebras which satisfy a given set of identities
- operators on a class $\mathcal{K}$ of algebras
    - **H**
    - **S**
    - **P**

Monounary algebras contra (some) other algebras: Pseudovarieties

- **variety** $=$ a class of those algebras which satisfy a given set of identities
- operators on a class $\mathcal{K}$ of algebras
  - **H**
  - **S**
  - **P**

Theorem

(Birkhoff) Each variety can be characterized as a class which is closed with respect to the operators **H**, **S**, **P**.

Pseudovarieties

Pseudovarieties

- operators on a class $\mathcal{K}$ of FINITE algebras

Pseudovarieties

- operators on a class $\mathcal{K}$ of FINITE algebras
  - **H**

Pseudovarieties

- operators on a class $\mathcal{K}$ of FINITE algebras
  - **H**
  - **S**

Pseudovarieties

- operators on a class $\mathcal{K}$ of FINITE algebras
  - $\mathbf{H}$
  - $\mathbf{S}$
  - $\mathbf{P}_F =$ forming direct products of finitely many members of $\mathcal{K}$

Pseudovarieties

- operators on a class $\mathcal{K}$ of FINITE algebras
  - **H**
  - **S**
  - $\mathbf{P}_F$ $=$ forming direct products of finitely many members of $\mathcal{K}$

- **pseudovariety** $=$ a class of finite algebras, which is closed with respect to the operators **H**, **S**, $\mathbf{P}_F$

Pseudovarieties

Pseudovarieties

Theorem

The smallest pseudovariety containing a class $\mathcal{K}$ of algebras of a given type is equal to $\mathbf{HSP}_F\mathcal{K}$;

Pseudovarieties

Theorem

The smallest pseudovariety containing
a class $\mathcal{K}$ *of algebras of a given type*
is equal to $\mathbf{HSP}_F\mathcal{K}$*;*

this class is called a **pseudovariety generated by** $\mathcal{K}$.

Pseudovarieties

Theorem

The smallest pseudovariety containing a class $\mathcal{K}$ of algebras of a given type is equal to $\mathbf{HSP}_F\mathcal{K}$*;*

this class is called a **pseudovariety generated by** $\mathcal{K}$.

pseudovariety $\mathcal{P}$

- **equational**, if there is a variety $\mathcal{V}$ such that $\mathcal{P}$ consists of all finite members of $\mathcal{V}$

Pseudovarieties

Theorem

The smallest pseudovariety containing
a class $\mathcal{K}$ of algebras of a given type
is equal to $\mathbf{HSP}_F\mathcal{K}$*;*

this class is called a **pseudovariety generated by** $\mathcal{K}$.

pseudovariety $\mathcal{P}$

- **equational**, if there is a variety $\mathcal{V}$ such that $\mathcal{P}$ consists of all finite members of $\mathcal{V}$
- **finitely generated**, if there exists a finite set of algebras generating $\mathcal{P}$

Pseudovarieties

Theorem

The smallest pseudovariety containing a class $\mathcal{K}$ of algebras of a given type is equal to $\mathbf{HSP}_F\mathcal{K}$;

this class is called a **pseudovariety generated by** $\mathcal{K}$.

pseudovariety $\mathcal{P}$

- **equational**, if there is a variety $\mathcal{V}$ such that $\mathcal{P}$ consists of all finite members of $\mathcal{V}$
- **finitely generated**, if there exists a finite set of algebras generating $\mathcal{P}$

Pseudovarieties

Theorem

The smallest pseudovariety containing a class $\mathcal{K}$ of algebras of a given type is equal to $\mathbf{HSP}_F\mathcal{K}$;

this class is called a **pseudovariety generated by** $\mathcal{K}$.

pseudovariety $\mathcal{P}$

- **equational**, if there is a variety $\mathcal{V}$ such that $\mathcal{P}$ consists of all finite members of $\mathcal{V}$
- **finitely generated**, if there exists a finite set of algebras generating $\mathcal{P}$

Theorem

Each finitely generated pseudovariety is equational.

Pseudovarieties

Pseudovarieties

(A, f) - finite *connected* monounary algebra with a cycle C

- **range** $r(A, f) = |C|$

Pseudovarieties

(A, f) - finite *connected* monounary algebra with a cycle C

- **range** $r(A, f) = |C|$
- **depth**
  $d(A, f) = \max\{\min\{m \in \mathbb{N} \cup \{0\} : f^m(x) \in C\} : x \in A\}$

Pseudovarieties

(A, f) - finite *connected* monounary algebra with a cycle C

- **range** $r(A, f) = |C|$
- **depth**
  $d(A, f) = \max\{\min\{m \in \mathbb{N} \cup \{0\} : f^m(x) \in C\} : x \in A\}$
- **width** $w(A, f) = \max\{|f^{-1}(x) \setminus C| : x \in A\}$

Pseudovarieties

(A, f) - finite *connected* monounary algebra with a cycle C

- **range** $r(A, f) = |C|$
- **depth**
  $d(A, f) = \max\{\min\{m \in \mathbb{N} \cup \{0\} : f^m(x) \in C\} : x \in A\}$
- **width** $w(A, f) = \max\{|f^{-1}(x) \setminus C| : x \in A\}$

Pseudovarieties

(A, f) - finite *connected* monounary algebra with a cycle C

- **range** $r(A, f) = |C|$
- **depth** $d(A, f) = \max\{\min\{m \in \mathbb{N} \cup \{0\} : f^m(x) \in C\} : x \in A\}$
- **width** $w(A, f) = \max\{|f^{-1}(x) \setminus C| : x \in A\}$

(A, f) - finite monounary algebra with the system of connected components $A_1, A_2, \dots, A_s$

- $r(A, f) = \mathrm{l.c.m.}\{r(A_\tau, f) : \tau \in \{1, 2, \dots, s\}\}$

Pseudovarieties

(A, f) - finite *connected* monounary algebra with a cycle C

- **range** $r(A, f) = |C|$
- **depth**
  $d(A, f) = \max\{\min\{m \in \mathbb{N} \cup \{0\} : f^m(x) \in C\} : x \in A\}$
- **width** $w(A, f) = \max\{|f^{-1}(x) \setminus C| : x \in A\}$

(A, f) - finite monounary algebra with the system of connected components $A_1, A_2, \ldots, A_s$

- $r(A, f) = \mathrm{l.c.m.}\{r(A_\tau, f) : \tau \in \{1, 2, \ldots, s\}\}$
- $d(A, f) = \max\{d(A_\tau, f) : \tau \in \{1, 2, \ldots, s\}\}$

Pseudovarieties

(A, f) - finite *connected* monounary algebra with a cycle C

- **range** $r(A, f) = |C|$
- **depth**
  $d(A, f) = \max\{\min\{m \in \mathbb{N} \cup \{0\} : f^m(x) \in C\} : x \in A\}$
- **width** $w(A, f) = \max\{|f^{-1}(x) \setminus C| : x \in A\}$

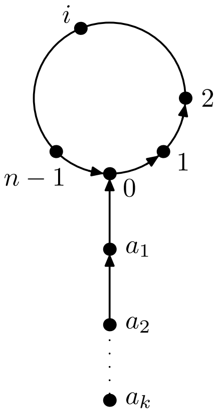
(A, f) - finite monounary algebra with the system of connected components $A_1, A_2, \ldots, A_s$

- $r(A, f) = \mathrm{l.c.m.}\{r(A_\tau, f) : \tau \in \{1, 2, \ldots, s\}\}$
- $d(A, f) = \max\{d(A_\tau, f) : \tau \in \{1, 2, \ldots, s\}\}$
- $w(A, f) = \max\{w(A_\tau, f) : \tau \in \{1, 2, \ldots, s\}\}$

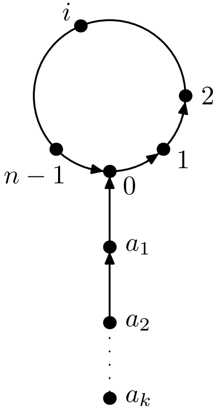
Pseudovarieties

- Make a characterization of finitely generated pseudovarieties of monounary algebras by finding algebras which generate them

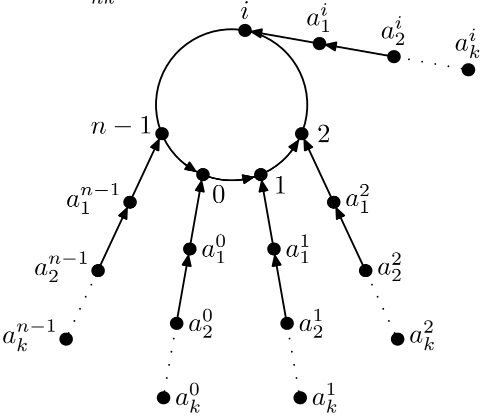
A_{nk}



A_{nk}


i
2
1
0
$n-1$
a_1
a_2
a_k

B_{nk}


i
a_1^i
a_2^i
a_k^i
$n-1$
2
0
1
a_1^{n-1}
a_2^{n-1}
a_k^{n-1}
a_1^0
a_2^0
a_k^0
a_1^1
a_2^1
a_k^1
a_1^2
a_2^2
a_k^2

D_{nkm}

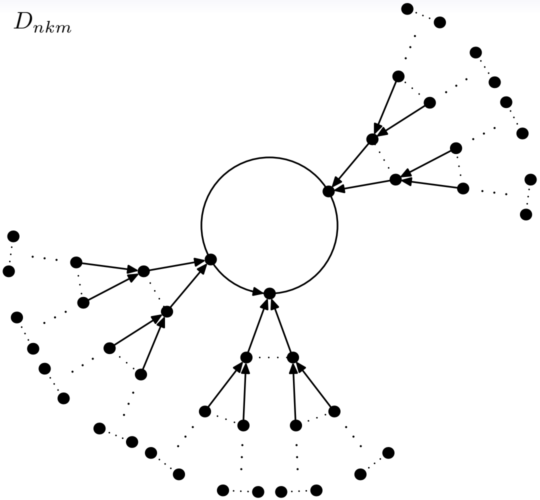


Figure:

Pseudovarieties

Pseudovarieties

(A'_{1k}, f) - a monounary algebra which is a disjoint union of

- (A_{1k}, f)
  and
- a one-element cycle $\{0'\}$ (a loop)

Pseudovarieties

(A'_{1k}, f) - a monounary algebra which is a disjoint union of

- (A_{1k}, f)
  and
- a one-element cycle $\{0'\}$ (a loop)

Pseudovarieties

$k \in \mathbb{N} \cup \{0\}$, $n, m, s \in \mathbb{N}$, $n > 1$

Pseudovarieties

$k \in \mathbb{N} \cup \{0\}$, $n, m, s \in \mathbb{N}$, $n > 1$

$$A_{nk} \quad \overset{\mathbf{P}_F, \mathbf{S}}{\Longrightarrow} \quad B_{nk}$$

Pseudovarieties

$k \in \mathbb{N} \cup \{0\}$, $n, m, s \in \mathbb{N}$, $n > 1$

$$A_{nk} \quad \stackrel{\mathbf{P}_F,\mathbf{S}}{\Longrightarrow} \quad B_{nk} \quad \stackrel{\mathbf{P}_F,\mathbf{S}}{\Longrightarrow} \quad D_{nkm}$$

Pseudovarieties

$k \in \mathbb{N} \cup \{0\}$, $n, m, s \in \mathbb{N}$, $n > 1$

$$A_{nk} \overset{\mathbf{P}_F, \mathbf{S}}{\Longrightarrow} B_{nk} \overset{\mathbf{P}_F, \mathbf{S}}{\Longrightarrow} D_{nkm} \overset{\mathbf{P}_F, \mathbf{S}}{\Longrightarrow} s \cdot D_{nkm}$$

Pseudovarieties

$k \in \mathbb{N} \cup \{0\}$, $n, m, s \in \mathbb{N}$, $n > 1$

$$A_{nk} \stackrel{\mathbf{P}_F,\mathbf{S}}{\Longrightarrow} B_{nk} \stackrel{\mathbf{P}_F,\mathbf{S}}{\Longrightarrow} D_{nkm}$$

$$\stackrel{\mathbf{P}_F,\mathbf{S}}{\Longrightarrow} s \cdot D_{nkm} \stackrel{\mathbf{H},\mathbf{S}}{\Longrightarrow} (A, f)$$

Pseudovarieties

$k \in \mathbb{N} \cup \{0\}$, $n, m, s \in \mathbb{N}$, $n > 1$

$$A_{nk} \xRightarrow{\mathbf{P}_F, \mathbf{S}} B_{nk} \xRightarrow{\mathbf{P}_F, \mathbf{S}} D_{nkm}$$
$$\xRightarrow{\mathbf{P}_F, \mathbf{S}} s \cdot D_{nkm} \xRightarrow{\mathbf{H}, \mathbf{S}} (A, f)$$

(A, f)

- $r(A, f)$ divides n
- $d(A, f) \leq k$
- $w(A, f) \leq m$
- (A, f) has at most s connected components

Pseudovarieties

$k \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$

Pseudovarieties

$k \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$

$$A_{1k} \quad \xrightarrow{\mathbf{P}_F, \mathbf{S}} \quad B_{1k}$$

Pseudovarieties

$k \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$

$$A_{1k} \quad \xrightarrow{\mathbf{P}_F, \mathbf{S}} \quad B_{1k}$$
$$\xrightarrow{\mathbf{P}_F, \mathbf{S}} \quad D_{1km}$$

Pseudovarieties

$k \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$

$$\begin{array}{rcccl} & A_{1k} & \overset{\mathbf{P}_F,\mathbf{S}}{\Longrightarrow} & B_{1k} & \\ \overset{\mathbf{P}_F,\mathbf{S}}{\Longrightarrow} & D_{1km} & \overset{\mathbf{S}}{\Longrightarrow} & (A, f) & \end{array}$$

Pseudovarieties

$k \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$

$$A_{1k} \overset{\mathbf{P}_F,\mathbf{S}}{\Longrightarrow} B_{1k}$$
$$\overset{\mathbf{P}_F,\mathbf{S}}{\Longrightarrow} D_{1km} \overset{\mathbf{S}}{\Longrightarrow} (A, f)$$

(A, f)

- (A, f) is connected
- $r(A, f) = 1$
- $d(A, f) \leq k$
- $w(A, f) \leq m$

Pseudovarieties

$k \in \mathbb{N} \cup \{0\}$, $m, s \in \mathbb{N}$

Pseudovarieties

$k \in \mathbb{N} \cup \{0\}$, $m, s \in \mathbb{N}$

$$A'_{1k} \quad \xrightarrow{\mathbf{P}_F, \mathbf{S}} \quad 2 \cdot D_{1km}$$

Pseudovarieties

$k \in \mathbb{N} \cup \{0\}$, $m, s \in \mathbb{N}$

$$A'_{1k} \overset{\mathbf{P}_F, \mathbf{S}}{\Longrightarrow} 2 \cdot D_{1km}$$
$$\overset{\mathbf{P}_F, \mathbf{S}}{\Longrightarrow} s \cdot D_{1km}$$

Pseudovarieties

$k \in \mathbb{N} \cup \{0\}$, $m, s \in \mathbb{N}$

$$
\begin{aligned}
A'_{1k} \quad &\xRightarrow{\mathbf{P}_F, \mathbf{S}} \quad 2 \cdot D_{1km} \\
\xRightarrow{\mathbf{P}_F, \mathbf{S}} \quad s \cdot D_{1km} \quad &\xRightarrow{\mathbf{S}} \quad (A, f)
\end{aligned}
$$

Pseudovarieties

$k \in \mathbb{N} \cup \{0\}$, $m, s \in \mathbb{N}$

$$A'_{1k} \overset{\mathbf{P}_F,\mathbf{S}}{\Longrightarrow} 2 \cdot D_{1km}$$
$$\overset{\mathbf{P}_F,\mathbf{S}}{\Longrightarrow} s \cdot D_{1km} \overset{\mathbf{S}}{\Longrightarrow} (A, f)$$

(A, f)

- $r(A, f) = 1$
- $d(A, f) \leq k$
- $w(A, f) \leq m$
- (A, f) has at most s connected components

Finitely generated pseudovarieties

Finitely generated pseudovarieties

- characterization of finitely generated pseudovarieties of monounary algebras by finding algebras which generate them

Finitely generated pseudovarieties

- characterization of finitely generated pseudovarieties of monounary algebras by finding algebras which generate them

Lemma

Let $k \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, $n > 1$. The pseudovariety generated by (A_{nk}, f) consists of all finite monounary algebras (A, f) such that

Finitely generated pseudovarieties

- characterization of finitely generated pseudovarieties of monounary algebras by finding algebras which generate them

Lemma

Let $k \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, $n > 1$. *The pseudovariety generated by* (A_{nk}, f) *consists of all finite monounary algebras* (A, f) *such that*

(i) $r(A, f)$ *divides* n,

(ii) $d(A, f) \leq k$.

Finitely generated pseudovarieties

Finitely generated pseudovarieties

Lemma

Let $k \in \mathbb{N} \cup \{0\}$. The pseudovariety generated by (A_{1k}, f) consists of all finite monounary algebras (A, f) with the following properties:

Finitely generated pseudovarieties

Lemma

Let $k \in \mathbb{N} \cup \{0\}$. The pseudovariety generated by (A_{1k}, f) consists of all finite monounary algebras (A, f) with the following properties:

(i) (A, f) *is connected,*

(ii) $r(A, f) = 1$,

(iii) $d(A, f) \leq k$.

Finitely generated pseudovarieties

Lemma

Let $k \in \mathbb{N} \cup \{0\}$. The pseudovariety generated by (A_{1k}, f) consists of all finite monounary algebras (A, f) with the following properties:

(i) *(A, f) is connected,*

(ii) $r(A, f) = 1$,

(iii) $d(A, f) \leq k$.

Lemma

Let $k \in \mathbb{N} \cup \{0\}$. The pseudovariety generated by (A'_{1k}, f) consists of all finite monounary algebras (A, f) such that

Finitely generated pseudovarieties

Lemma

Let $k \in \mathbb{N} \cup \{0\}$. The pseudovariety generated by (A_{1k}, f) consists of all finite monounary algebras (A, f) with the following properties:

(i) (A, f) *is connected,*

(ii) $r(A, f) = 1$,

(iii) $d(A, f) \leq k$.

Lemma

Let $k \in \mathbb{N} \cup \{0\}$. The pseudovariety generated by (A'_{1k}, f) consists of all finite monounary algebras (A, f) such that

(i) $r(A, f) = 1$,

(ii) $d(A, f) \leq k$.

Finitely generated pseudovarieties

Finitely generated pseudovarieties

Theorem

A class $\mathcal{P}$ of finite monounary algebras is a finitely generated pseudovariety if and only if one of the following conditions is satisfied:

Finitely generated pseudovarieties

Theorem

A class $\mathcal{P}$ of finite monounary algebras is a finitely generated pseudovariety if and only if one of the following conditions is satisfied:

(a) *There are $k \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ such that $\mathcal{P}$ consists of all finite monounary algebras (A, f) with*

Finitely generated pseudovarieties

Theorem

A class $\mathcal{P}$ of finite monounary algebras is a finitely generated pseudovariety if and only if one of the following conditions is satisfied:

(a) *There are $k \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ such that $\mathcal{P}$ consists of all finite monounary algebras (A, f) with*

  (i) *$r(A, f)$ divides n,*

  (ii) *$d(A, f) \leq k$.*

Finitely generated pseudovarieties

Theorem

A class $\mathcal{P}$ of finite monounary algebras is a finitely generated pseudovariety if and only if one of the following conditions is satisfied:

(a) *There are $k \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ such that $\mathcal{P}$ consists of all finite monounary algebras (A, f) with*
  (i) *$r(A, f)$ divides n,*
  (ii) *$d(A, f) \leq k$.*

(b) *There is $k \in \mathbb{N} \cup \{0\}$ such that $\mathcal{P}$ consists of all finite monounary algebras (A, f) with the following properties:*

Finitely generated pseudovarieties

Theorem

A class $\mathcal{P}$ of finite monounary algebras is a finitely generated pseudovariety if and only if one of the following conditions is satisfied:

(a) *There are $k \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ such that $\mathcal{P}$ consists of all finite monounary algebras (A, f) with*
  - (i) $r(A, f)$ *divides* n,
  - (ii) $d(A, f) \leq k$.

(b) *There is $k \in \mathbb{N} \cup \{0\}$ such that $\mathcal{P}$ consists of all finite monounary algebras (A, f) with the following properties:*
  - (i) (A, f) *is connected,*
  - (ii) $r(A, f) = 1$,
  - (iii) $d(A, f) \leq k$.

Finitely generated pseudovarieties

Theorem

A class $\mathcal{P}$ of finite monounary algebras is a finitely generated pseudovariety if and only if one of the following conditions is satisfied:

(a) *There are $k \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ such that $\mathcal{P}$ consists of all finite monounary algebras (A, f) with*
   (i) $r(A, f)$ *divides* n,
   (ii) $d(A, f) \leq k$.

(b) *There is $k \in \mathbb{N} \cup \{0\}$ such that $\mathcal{P}$ consists of all finite monounary algebras (A, f) with the following properties:*
   (i) (A, f) *is connected,*
   (ii) $r(A, f) = 1$,
   (iii) $d(A, f) \leq k$.

If (a), $n > 1$, *then $\mathcal{P}$ is generated by (A_{nk}, f) and if $n = 1$, then by* (A'_{1k}, f).

Finitely generated pseudovarieties

Theorem

A class $\mathcal{P}$ of finite monounary algebras is a finitely generated pseudovariety if and only if one of the following conditions is satisfied:

(a) *There are $k \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ such that $\mathcal{P}$ consists of all finite monounary algebras (A, f) with*
   - (i) $r(A, f)$ *divides* n,
   - (ii) $d(A, f) \leq k$.

(b) *There is $k \in \mathbb{N} \cup \{0\}$ such that $\mathcal{P}$ consists of all finite monounary algebras (A, f) with the following properties:*
   - (i) (A, f) *is connected,*
   - (ii) $r(A, f) = 1$,
   - (iii) $d(A, f) \leq k$.

If (a), $n > 1$, *then $\mathcal{P}$ is generated by (A_{nk}, f) and if $n = 1$, then by (A'_{1k}, f). If* (b), *$\mathcal{P}$ is generated by (A_{1k}, f).*

Finitely generated and equational pseudovarieties

Finitely generated and equational pseudovarieties

- $\mathcal{U}$ - the system of all monounary algebras
- $\mathcal{U}_F$ - the system of all finite monounary algebras

Finitely generated and equational pseudovarieties

- $\mathcal{U}$ - the system of all monounary algebras
- $\mathcal{U}_F$ - the system of all finite monounary algebras

Theorem

A class $\mathcal{P}$ of finite monounary algebras is a finitely generated pseudovariety if and only if $\mathcal{P}$ consists of all finite algebras belonging to a variety $\mathcal{V} \neq \mathcal{U}$ of monounary algebras.

Finitely generated and equational pseudovarieties

- $\mathcal{U}$ - the system of all monounary algebras
- $\mathcal{U}_F$ - the system of all finite monounary algebras

Theorem

A class $\mathcal{P}$ of finite monounary algebras is a finitely generated pseudovariety if and only if $\mathcal{P}$ consists of all finite algebras belonging to a variety $\mathcal{V} \neq \mathcal{U}$ of monounary algebras.

Corollary

If $\mathcal{P} \neq \mathcal{U}_F$ is a pseudovariety of monounary algebras, then it is finitely generated if and only if it is equational.

Pseudovarieties

Pseudovarieties

$\mathcal{P}$ - pseudovariety

Pseudovarieties

$\mathcal{P}$ - pseudovariety

- generated by algebras (S_m, f), $m \in \mathbb{N}$ of types (A_{nk}, f) or (A'_{1k}, f)

Pseudovarieties

$\mathcal{P}$ - pseudovariety

- generated by algebras (S_m, f), $m \in \mathbb{N}$ of types (A_{nk}, f) or (A'_{1k}, f)
- denote $\mathcal{P}_n$ - pseudovariety, generated by the set $\{(S_m, f) : m \leq n\}$

Pseudovarieties

$\mathcal{P}$ - pseudovariety

- generated by algebras (S_m, f), $m \in \mathbb{N}$ of types (A_{nk}, f) or (A'_{1k}, f)
- denote $\mathcal{P}_n$ - pseudovariety, generated by the set $\{(S_m, f) : m \leq n\}$

Then

$$\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n$$

Pseudovarieties

$\mathcal{P}$ - pseudovariety

- generated by algebras (S_m, f), $m \in \mathbb{N}$ of types (A_{nk}, f) or (A'_{1k}, f)
- denote $\mathcal{P}_n$ - pseudovariety, generated by the set $\{(S_m, f) : m \leq n\}$

Then

$$\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n$$

$$r(\mathcal{P}) = \begin{cases} \mathrm{l.c.m.}\{r(S_m, f) : m \in \mathbb{N}\} & \text{if it exists,} \\ \infty & \text{otherwise} \end{cases}$$

Pseudovarieties

$\mathcal{P}$ - pseudovariety

- generated by algebras (S_m, f), $m \in \mathbb{N}$ of types (A_{nk}, f) or (A'_{1k}, f)
- denote $\mathcal{P}_n$ - pseudovariety, generated by the set $\{(S_m, f) : m \leq n\}$

Then

$$\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n$$

$$r(\mathcal{P}) = \begin{cases} \mathrm{l.c.m.}\{r(S_m, f) : m \in \mathbb{N}\} & \text{if it exists,} \\ \infty & \text{otherwise} \end{cases}$$

$$d(\mathcal{P}) = \begin{cases} \max\{d(S_m, f) : m \in \mathbb{N}\} & \text{if it exists,} \\ \infty & \text{otherwise} \end{cases}$$

Pseudovarieties

Pseudovarieties

Theorem

(i) $n = r(\mathcal{P}) \neq \infty$, $k = d(\mathcal{P}) \neq \infty \implies \mathcal{P}$ *is finitely generated*

Pseudovarieties

Theorem

(i) $n = r(\mathcal{P}) \neq \infty$, $k = d(\mathcal{P}) \neq \infty \implies \mathcal{P}$ *is finitely generated*

(ii) $n = r(\mathcal{P}) \neq \infty$, $d(\mathcal{P}) = \infty \implies$

- $n = 1$, $\mathcal{P}$ *consists of all finite connected algebras with a one-element cycle, or*
- $\mathcal{P}$ *consists of all finite algebras* (A, f) *such that* $r(A, f)$ *divides* n

Pseudovarieties

Theorem

(i) $n = r(\mathcal{P}) \neq \infty,\ k = d(\mathcal{P}) \neq \infty \implies \mathcal{P}$ *is finitely generated*

(ii) $n = r(\mathcal{P}) \neq \infty,\ d(\mathcal{P}) = \infty \implies$

- $n = 1$, $\mathcal{P}$ *consists of all finite connected algebras with a one-element cycle, or*
- $\mathcal{P}$ *consists of all finite algebras* (A, f) *such that* $r(A, f)$ *divides* n

(iii) $r(\mathcal{P}) = \infty,\ k = d(\mathcal{P}) \neq \infty \implies \mathcal{P}$ *consists of all finite algebras* (A, f) *such that* $r(A, f)$ *divides* $r(S_m, f)$ *for some* $m \in \mathbb{N}$ *and* $d(A, f) \leq k$

Pseudovarieties

Theorem

(i) $n = r(\mathcal{P}) \neq \infty,\ k = d(\mathcal{P}) \neq \infty \implies \mathcal{P}$ *is finitely generated*

(ii) $n = r(\mathcal{P}) \neq \infty,\ d(\mathcal{P}) = \infty \implies$

- $n = 1$, $\mathcal{P}$ *consists of all finite connected algebras with a one-element cycle, or*
- $\mathcal{P}$ *consists of all finite algebras* (A, f) *such that* $r(A, f)$ *divides* n

(iii) $r(\mathcal{P}) = \infty,\ k = d(\mathcal{P}) \neq \infty \implies \mathcal{P}$ *consists of all finite algebras* (A, f) *such that* $r(A, f)$ *divides* $r(S_m, f)$ *for some* $m \in \mathbb{N}$ *and* $d(A, f) \leq k$

(iv) $r(\mathcal{P}) = d(\mathcal{P}) = \infty \implies \mathcal{P}$ *consists of all finite algebras* (A, f) *such that* $r(A, f)$ *divides* $r(S_m, f)$ *for some* $m \in \mathbb{N}$.

Monounary algebras contra (some) other algebras: Globals

Monounary algebras contra (some) other algebras: Globals

- $P(\mathcal{A}) = (P(A), F)$ is a *global* of an algebra $\mathcal{A} = (A, F)$ if $f \in F$ is n-ary, $B_1, \ldots, B_n$ are subsets of A, then

$$f(B_1, \ldots, B_n) = \{f(b_1, \ldots, b_n) : b_1 \in B_1, \ldots, b_n \in B_n\}.$$

Monounary algebras contra (some) other algebras: Globals

- $P(\mathcal{A}) = (P(A), F)$ is a *global* of an algebra $\mathcal{A} = (A, F)$ if $f \in F$ is n-ary, $B_1, \dots, B_n$ are subsets of A, then

$$f(B_1, \dots, B_n) = \{f(b_1, \dots, b_n) : b_1 \in B_1, \dots, b_n \in B_n\}.$$

- A class $\mathcal{C}$ of algebraic structures of the same type is said to be *globally determined* if for each $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{C}$

$$P(\mathcal{A}_1) \cong P(\mathcal{A}_2) \implies \mathcal{A}_1 \cong \mathcal{A}_2.$$

Globals

Globals

- class of all groups **is** globally determined (T. Tamura, J. Shafer 1967)

Globals

- class of all groups **is** globally determined (T. Tamura, J. Shafer 1967)
- class of all semigroups **fails to be** globally determined (E. M. Mogiljanskaja 1973)

Globals

- class of all groups **is** globally determined (T. Tamura, J. Shafer 1967)
- class of all semigroups **fails to be** globally determined (E. M. Mogiljanskaja 1973)
- class of all monounary algebras **fails to be** globally determined (A. Drápal 1985)

Globals

- class of all groups **is** globally determined (T. Tamura, J. Shafer 1967)
- class of all semigroups **fails to be** globally determined (E. M. Mogiljanskaja 1973)
- class of all monounary algebras **fails to be** globally determined (A. Drápal 1985)
- class of all finite monounary algebras **is** globally determined (A. Drápal 1985)

Globals

Globals

J. Herchl, DJS 2007:

Proposition

There exist 7-element algebras $\mathcal{A}_1, \mathcal{A}_2$ with two unary operations such that

(i) *each operation makes a cycle of the corresponding algebra,*

(ii) $\mathcal{A}_1 \not\cong \mathcal{A}_2$,

(iii) $P(\mathcal{A}_1) \cong P(\mathcal{A}_2)$.

Corollary

The class of all finite biunary algebras is not globally determined.

Theorem

The class of all finite unary algebras fails to be globally determined.

Figure: