Enforced hamiltonian cycles

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Podporujeme výskumné aktivity na Slovensku/
Projekt je spolufinancovaný zo zdrojov EÚ

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Proposition

$$\begin{array}{c} \text{2-hamiltonian} \\ \downarrow \\ \text{1-hamiltonian} \\ \downarrow \\ \text{hamiltonian} \\ \downarrow \\ \text{traceable} \end{array}$$

H-force number

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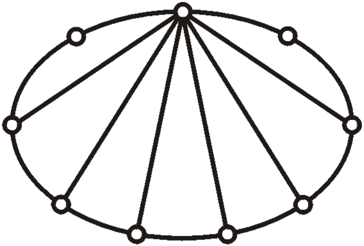
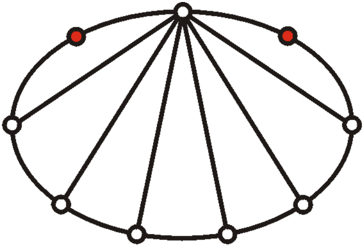
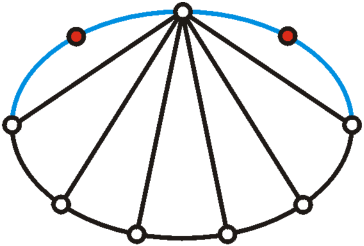
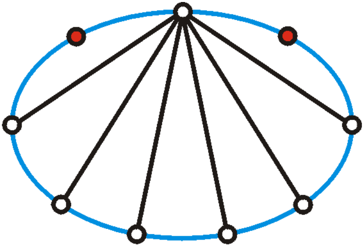
A nonempty set $X \subseteq V(G)$ is called a hamiltonian cycle enforcing set (**H-force set**) of G if every X-cycle of G is hamiltonian.

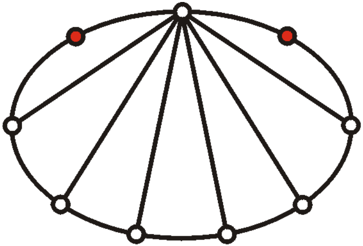
For a hamiltonian graph G we define **H-force number** $h = h(G)$ of G as the smallest cardinality of an H-force set of G.

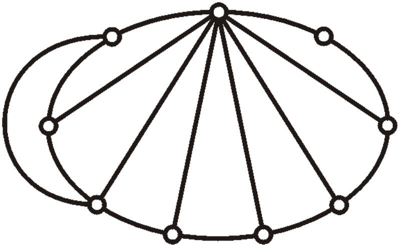


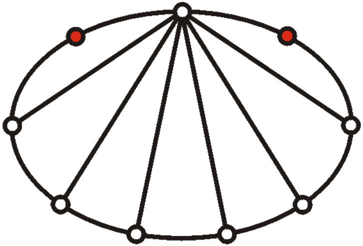


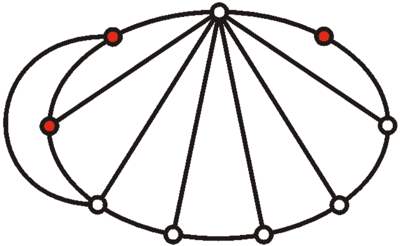


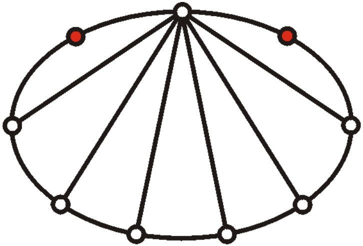


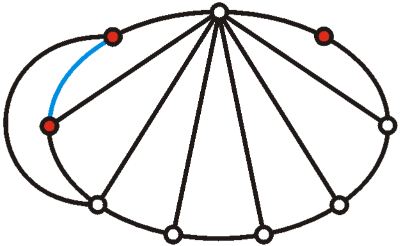


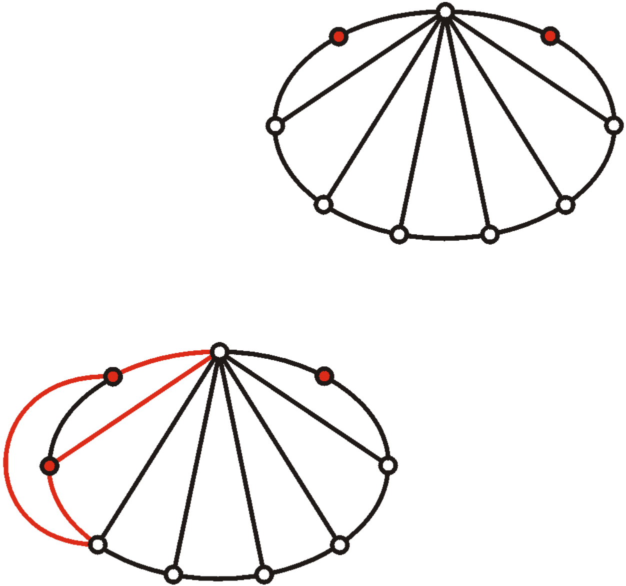


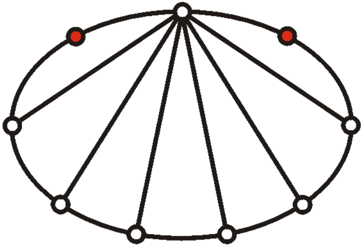


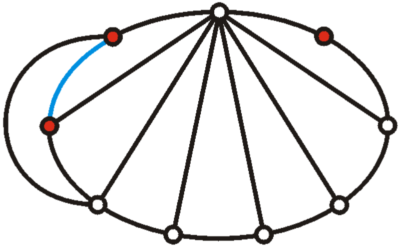


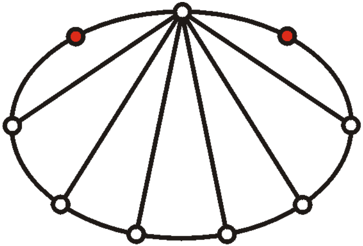


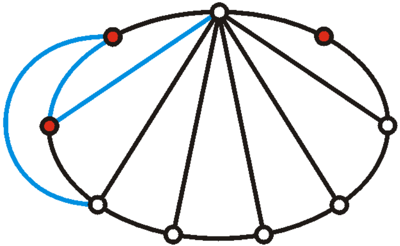


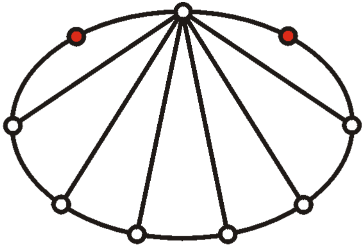


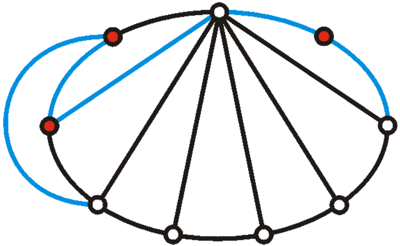


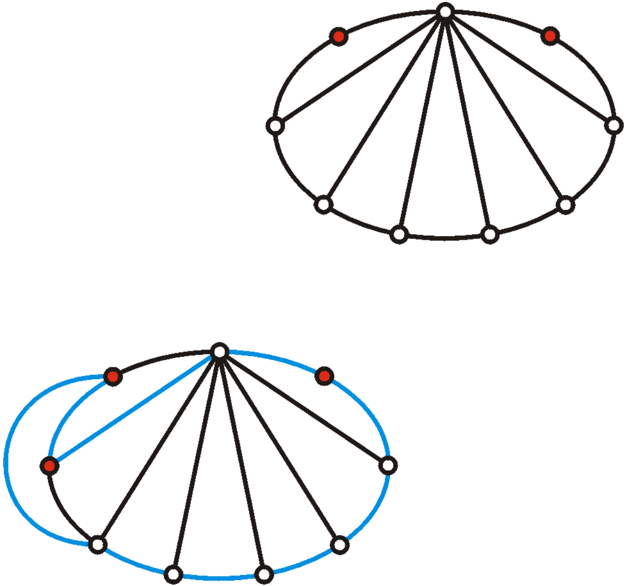


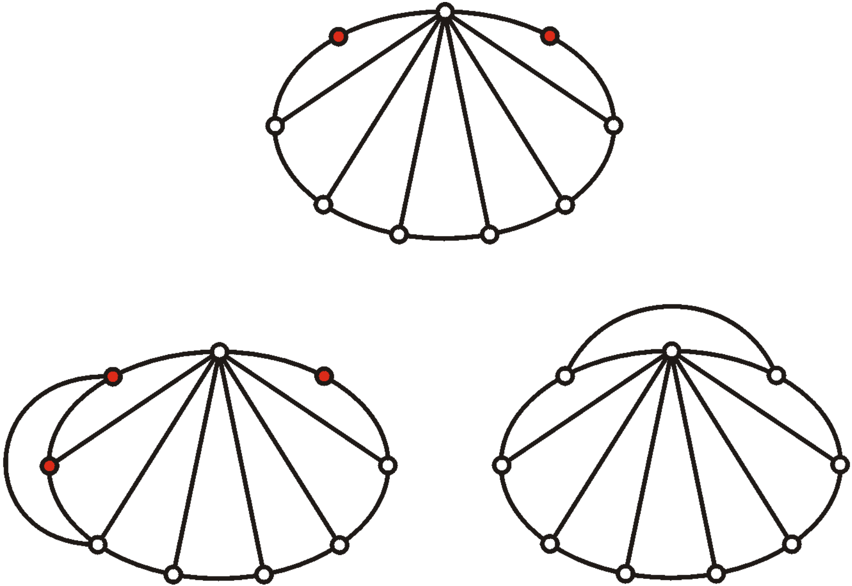


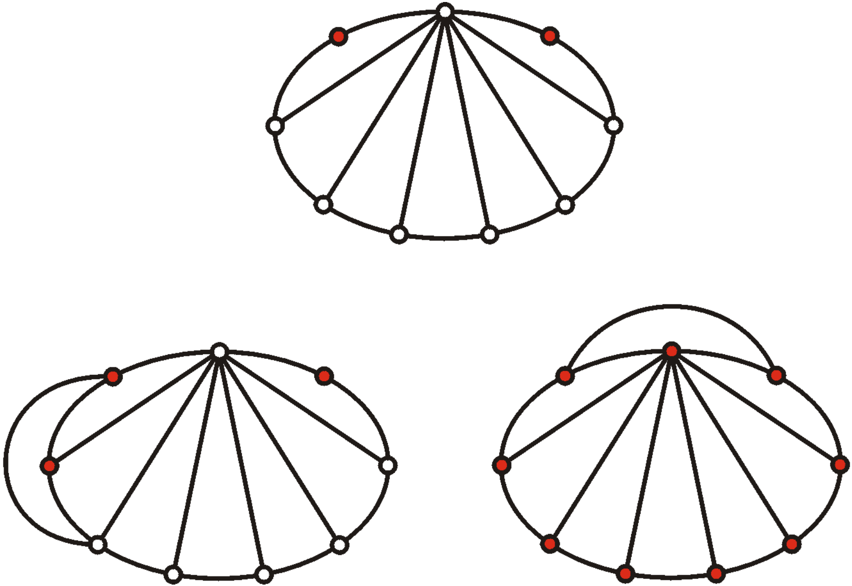












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If C is a nonhamiltonian cycle of G then any H-force set of G contains a vertex of $V(G) \setminus V(C)$.

Theorem (F., Hexel, Jendrol')

For all integers $n \geq 10$ and k where $1 \leq k \leq n$, there exists a hamiltonian graph G of order n with $h(G) = k$.

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Proposition

$h(G) = 1 \Leftrightarrow G = C_n$

1-hamiltonian graphs

The number of vertices (the order) of a graph will be denoted by n

Proposition

$h(G) = n \Leftrightarrow G$ 1-hamiltonian

Theorem (Chartrand, Kapoor, Link, 1970)

$\sigma_2(G) \geq n+1$ $(n \geq 4)$ $\Rightarrow$ G *1-hamiltonian*

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Theorem (Chartrand, Hobbs, Jung, Kapoor, Link, Nash-Williams, 1974)

G *2-connected* $\Rightarrow$ G^2 *1-hamiltonian*

Theorem (Ore, 1959)

$\sigma_2(G) \geq n+1 \ (n \geq 4) \ \Rightarrow \ G$ *hamiltonian-connected*

Theorem (Chvátal, Erdős, 1972)

$\alpha(G) < \kappa(G) \ (\kappa(G) \geq 3) \ \Rightarrow \ G$ *hamiltonian-connected*

Theorem (Thomassen, 1983, and Chiba, Nishizeki, 1986)

G *4-connected planar* $\Rightarrow \ G$ *hamiltonian-connected*

Theorem (Barefoot, 1987)

G *a Halin graph* $\Rightarrow \ G$ *hamiltonian-connected*

Theorem (Kanetkar, Rao, 1984)

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Theorem (Chartrand, Hobbs, Jung, Kapoor, Link, Nash-Williams, 1974)

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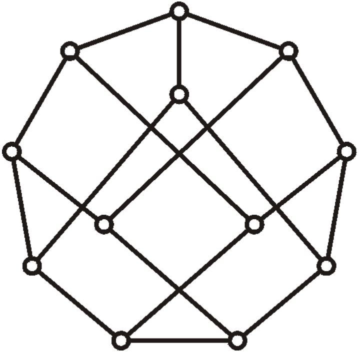

1-hamiltonian,
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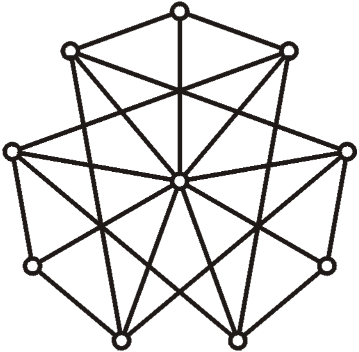

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Bipartite graphs, planar graphs

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Corollary

$h(K_{\frac{n}{2},\frac{n}{2}}) = \frac{n}{2}$

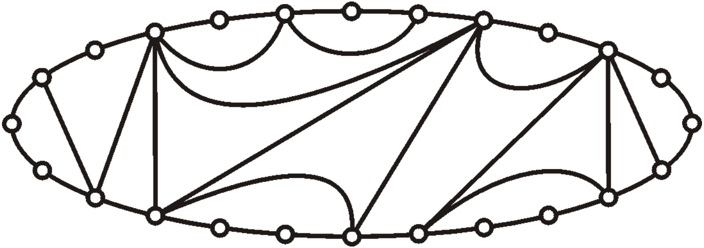
Theorem (F., Hexel, Jendrol')

If G is a hamiltonian graph with $\kappa(G) \geq 3$ then

$$h(G) \geq \begin{cases} \kappa(G), & \textit{if } G \textit{ bipartite} \\ \kappa(G) + 1, & \textit{if } G \textit{ non-bipartite} \end{cases}$$

Definition

The **weak-dual** $D^*(G)$ of an outerplanar graph G is its dual without the vertex corresponding to the outerface and $\ell(G)$ denotes the number of leafs of $D^*(G)$.



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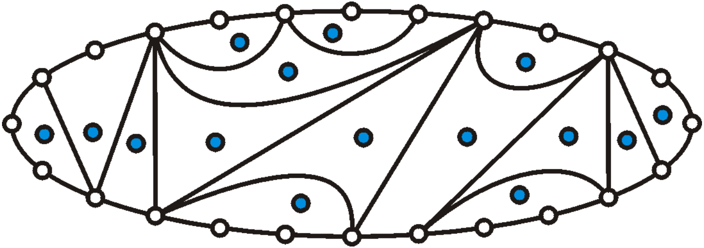

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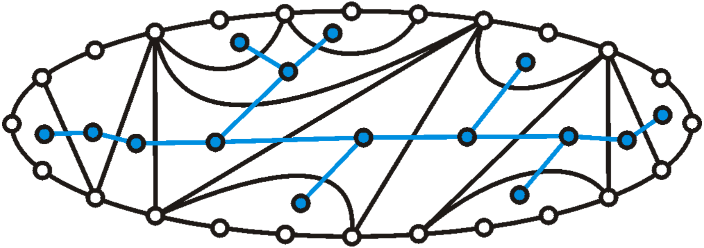

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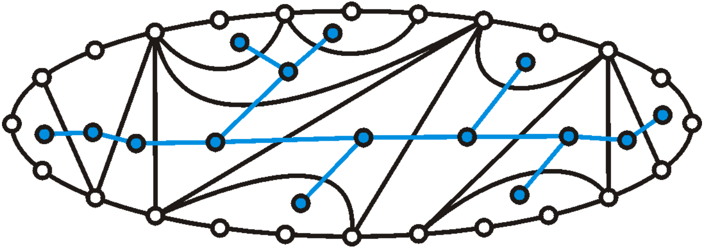


Theorem (F., Hexel, Jendrol')

If $G \neq C_n$ is an outerplanar hamiltonian graph then $h(G) = \ell(G)$.

For a plane graph G with a hamiltonian cycle C let G_1 (or G_2) be the outerplanar graph consisting of C and all edges of G lying inside (outside) of C.

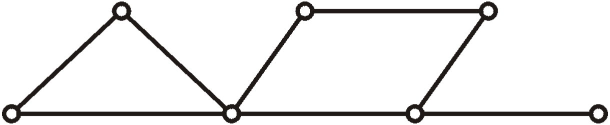
Theorem (F., Hexel, Jendrol')

If G is a planar hamiltonian graph with $\delta(G) \geq 3$ then $h(G) \geq \ell(G_1) + \ell(G_2) \geq 4$.

Prisms over graphs

Definition

The **prism over a graph** G is the Cartesian product $G \square K_2$



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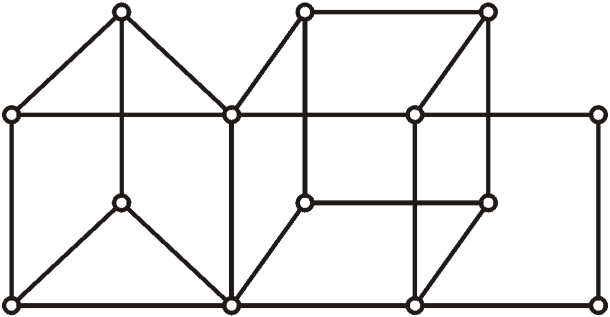

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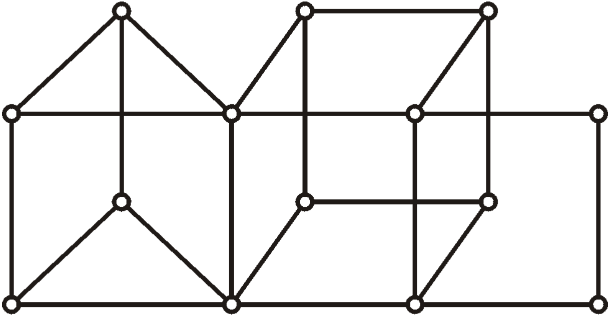

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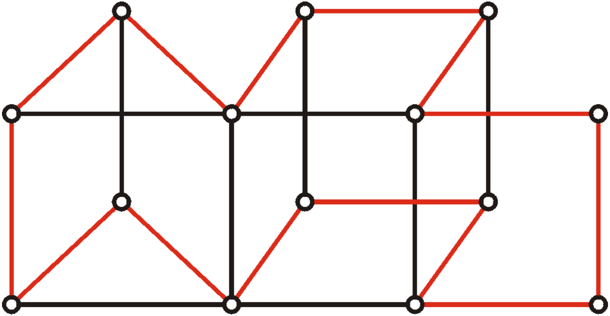

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Proposition

1-hamiltonian

$\downarrow$

hamiltonian

$\downarrow$

traceable

$\downarrow$

prism-hamiltonian

Let G be a graph of order $\frac{n}{2}$ (then the prism $G\Box K_2$ is of order n)

Theorem (F., Hexel, Jendrol')

If G is hamiltonian then $h(G\Box K_2) = \begin{cases} \frac{n}{2}, & \text{if } G \text{ bipartite} \\ n, & \text{if } G \text{ non-bipartite} \end{cases}$

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Theorem (F.)

If G is a traceable graph with a articulation points and p pendant vertices then $h(G \Box K_2) = \begin{cases} \frac{n}{2} - a, & \textit{if } G \textit{ bipartite} \\ n - 2a - p, & \textit{if } G \textit{ non-bipartite} \end{cases}$

Theorem (Fleischner, 1974)

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Theorem (Kaiser, Kráľ, Rosenfeld, Ryjáček, Voss, 2007)

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G connected $\Rightarrow$ G^2 prism-hamiltonian

Theorem (F.)

G connected $\Rightarrow$ $h(G^2 \square K_2) = n$

Theorem (Fleischner, 1974)

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Theorem (F.)

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Graphs with small H-force number

Proposition

$h(G) = 1 \Leftrightarrow G = C_n$

Definition

Let $C = [v_1, v_2, \ldots, v_n]$ be a hamiltonian cycle of G. A chord $v_i v_j$ $(i < j - 1)$ **separates** vertices v_k, v_l $(k < l - 1)$ on C, if they belong to different components of $C - v_i - v_j$, and, moreover, **crosses** the chord $v_k v_l$, if $v_k v_l \in E(G)$.

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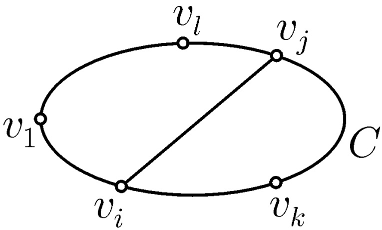

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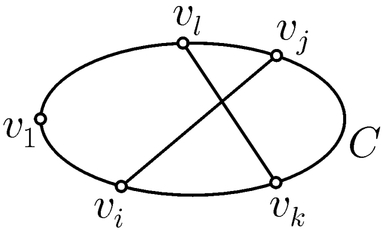

Theorem (F., Hexel, Jendrol')

Let $G \neq C_n$ be a hamiltonian graph and $C = [v_1, v_2, \ldots, v_n]$ be a hamiltonian cycle of G. Then $h(G) = 2$ if and only if

1. *there exist $x, y \in V(G)$, $\deg_G(x) = \deg_G(y) = 2$, such that every chord $v_i v_j$ ($i < j - 1$) separates x, y and*
2. *for every pair $v_i v_j$ ($i < j - 1$) and $v_k v_l$ ($k < l - 1$) of crossing chords $v_i v_k, v_j v_l \in E(C)$*

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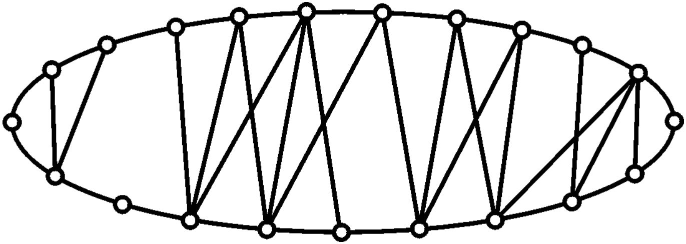

Theorem (F., Hexel, Jendrol')

Let $G \neq C_n$ be a hamiltonian graph and $C = [v_1, v_2, \ldots, v_n]$ be a hamiltonian cycle of G. Then $h(G) = 2$ if and only if

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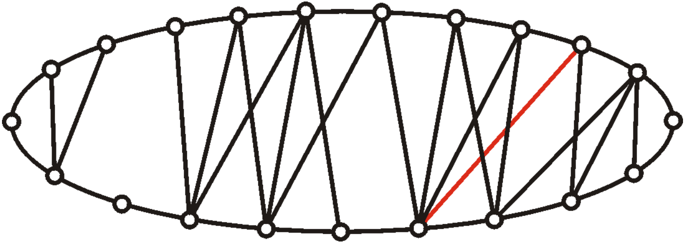

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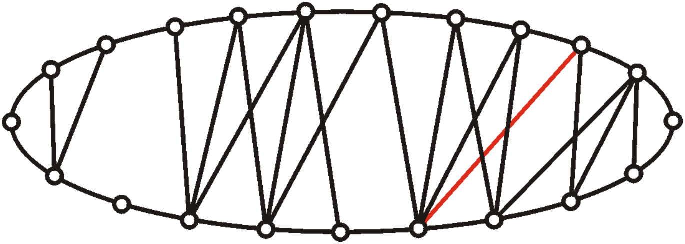

Corollary

$h(G) = 2 \Rightarrow \kappa(G) = 2$ *and G planar*

Theorem (F., Hexel, Jendrol')

If G is a 3-connected hamiltonian graph then

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If G *is a 3-connected hamiltonian graph then*

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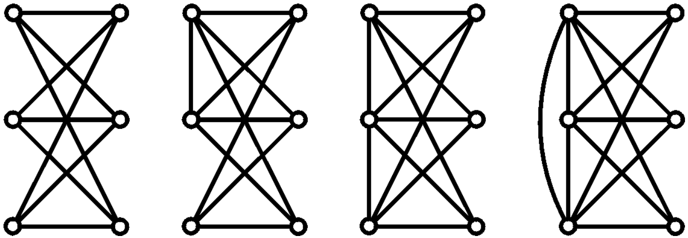

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Theorem (F., Hexel, Jendrol')

If G is a planar 3-connected hamiltonian graph then

- $h(G) \geq 5$ *or*
- $G = K_4$ *or*
- G *results from the graph of the cube by adding any edges in one partition class*

Thank you.