Constructions of Supermagic Graphs

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Pracovné stretnutie riešitel'ov Centra excelentnosti CEX CaKS

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Európska únia
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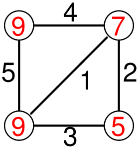
Index-mapping

Definition

Let G be a simple graph without isolated vertices and let f be a mapping from $E(G)$ into positive integers. The index-mapping of f is the mapping f^* from $V(G)$ into positive integers defined by

$$f^*(v) = \sum_{vu \in E(G)} f(vu) \quad \text{for every } v \in V(G)$$

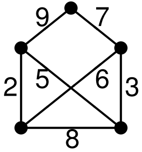


Magic labelling

Definition

An injective mapping f from $E(G)$ into positive integers is called a magic labelling of G for an index λ if its index-mapping f^* satisfies

$$f^*(v) = \lambda \quad \text{for all } v \in V(G).$$

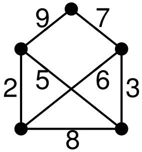


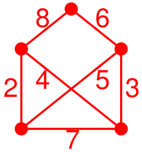
$\lambda = 16$

Supermagic labelling

definition

A magic labelling f of G is called a supermagic labelling if the set $\{f(e) : e \in E(G)\}$ consists of consecutive positive integers.





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A graph G is called supermagic (magic) whenever there exists a supermagic (magic) labelling of G.

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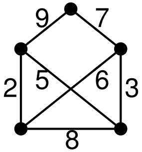

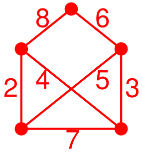

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History

Magic graphs

- **1963** – J. Sedláček: introduction of magic graphs
- **1978** – M. Doob: characterization of regular magic graphs
- **1983** – S. Jezný, M. Trenkler: characterization of all magic graphs
- **1988** – R. H. Jeurissen: other characterization of magic graphs

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Supermagic graphs

- **1966** – B. M. Stewart: introduction of supermagic graphs
- **1967** – B. M. Stewart: characterization of supermagic complete graphs
- **2000** – J. I.: characterization of supermagic complete multipartite graphs and supermagic cubes
- **2004** – J. I., Z. Lastivková, A. Semaničová: characterization of supermagic line graphs of regular bipartite graphs

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Sporadic constructions

Constructions used only for special graphs.

Theorem B. M. Stewart, 1967

The complete graph K_n is supermagic if and only if either $n \geq 6$ and $n \not\equiv 0 \pmod 4$ or $n = 2$.

Theorem J. Sedláček, 1976

The Möbius ladder M_p is supermagic graph for every odd integer $p \geq 3$.

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Graphs with supermagic factors

Constructions used for graphs decomposable into edge disjoint supermagic factors.

Theorem N. Hartsfield, G. Ringel, 1990

Let $F_1, F_2, \ldots, F_k$ be mutually edge-disjoint supermagic (regular) factors of a graph G which form its decomposition. Then G is supermagic.

Copies of supermagic graphs

Constructions used for regular supermagic graphs.

Theorem J. I., 2000

Let G be a supermagic regular graph decomposable into $k \geq 2$ edge-disjoint d-factors. Then it holds:

- if k is even, then mG is supermagic for every positive integer m;
- if k is odd, then mG is supermagic for every odd positive integer m.

Corollary J. I., Z. Lastivková, A. Semaničová, 2004

Let G be a bipartite d-regular graph, where $d \geq 3$. Then the line graph $L(G)$ is supermagic.

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Graphs decomposable into 2-factors

Theorem **N. Hartsfield, G. Ringel, 1990**

Let G be a bipartite 4-regular graph decomposable into two edge-disjoint Hamilton cycles. Then G is supermagic.

Theorem **J. I., 2000**

Let G be a bipartite 4-regular graph which can be decomposed into pairwise edge-disjoint 4-cycles. Then G is supermagic.

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Let G be a 3-regular graph containing a 1-factor. Then the line graph of G is a supermagic graph.

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Graphs decomposable into Eulerian factors

Theorem J. I., 2007

Let G be a $4k$-regular bipartite graph which can be decomposed into two edge-disjoint connected $2k$-factors. Then G is a supermagic graph.

Corollary

Let G be a $4k$-regular bipartite graph of order $2n$. If $4k-2 > n/2$, then G is a supermagic graph.

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Graphs decomposable into Eulerian factors

Theorem J. I., 2007

Let G be a d-regular bipartite graph of order $2n$ such that one of the following conditions is satisfied:

- $d \equiv 0 \pmod 4$ and $d - 2 > n/2$,
- $d \equiv 1 \pmod 4$, $n \equiv 1 \pmod 2$, $d - 11 > n/2$ and $d \geq (3n + 2)/4$,
- $d \equiv 2 \pmod 4$, $n \equiv 1 \pmod 2$ and $d - 8 > n/2$,
- $d \equiv 2 \pmod 4$, $n \equiv 0 \pmod 2$, $d - 8 > n/2$ and $d \geq (3n + 2)/4$,
- $d \equiv 3 \pmod 4$, $n \equiv 1 \pmod 2$, $d - 5 > n/2$ and $d \geq (3n + 2)/4$.

Then G is a supermagic graph.

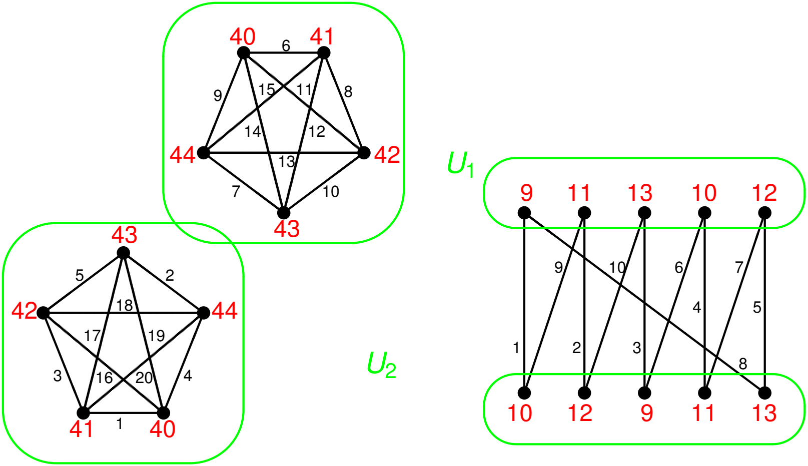
Double-consecutive labelling

Definition

Let U_1, U_2 be subsets of $V(G)$ such that $|U_1| = |U_2| = n$, $U_1 \cup U_2 = V(G)$ and $U_1 \cap U_2 = \emptyset$. An injective mapping f from $E(G)$ into positive integers is called a double-consecutive labelling (DC-labelling) with respect to (U_1, U_2) if its index-mapping f^* satisfies

$$f^*(U_1) = f^*(U_2) = \{a, a+1, \dots, a+n-1\} \quad \text{for some integer } a.$$

DC-labellings



DC-labellings of bipartite graphs

Lemma

Let G be a 1-regular bipartite graph of order $2n$ with parts U_1 and U_2. Then there is a DC-labelling f of G with respect to (U_1, U_2).

Lemma

Let G be a connected 2-regular bipartite graph of order $2n$ with parts U_1 and U_2. If n is an odd integer, then there is a DC-labelling f of G with respect to (U_1, U_2).

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DC-labellings of disjoint union of two graphs

Lemma

Let G_1 and G_2 be disjoint 3-regular Hamiltonian bipartite graphs each of order $n = 4k$, $k \geq 2$. Then there exists a DC-labelling f of $G_1 \cup G_2$ with respect to $(V(G_1), V(G_2))$.

Lemma

Let G_1 and G_2 be disjoint regular Hamiltonian graphs each of odd order n and degree 4 (6). Then there exists a DC-labelling f of $G_1 \cup G_2$ with respect to $(V(G_1), V(G_2))$.

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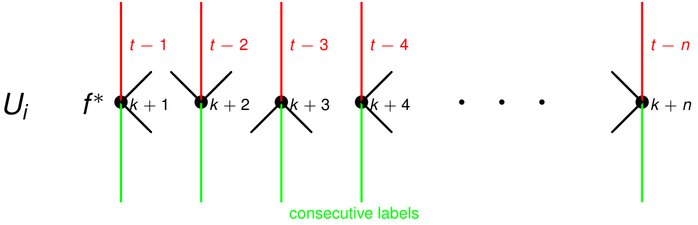
Extension of DC-labelling



Extension of DC-labelling

Lemma

Let H be a balanced bipartite graph of order $2n$ with parts U_1 and U_2. Let F be a 2-factor of H and let g be a DC-labelling of $G = H - F$ with respect to (U_1, U_2). Then there exists a DC-labeling f of H with respect to (U_1, U_2).

Lemma

Let H_1 and H_2 be disjoint graphs each of order n. Let F be a 4-factor of $H = H_1 \cup H_2$ and let g be a DC-labelling of $G = H - F$ with respect to $(V(H_1), V(H_2))$. Then there exists a DC-labeling f of H with respect to $(V(H_1), V(H_2))$.

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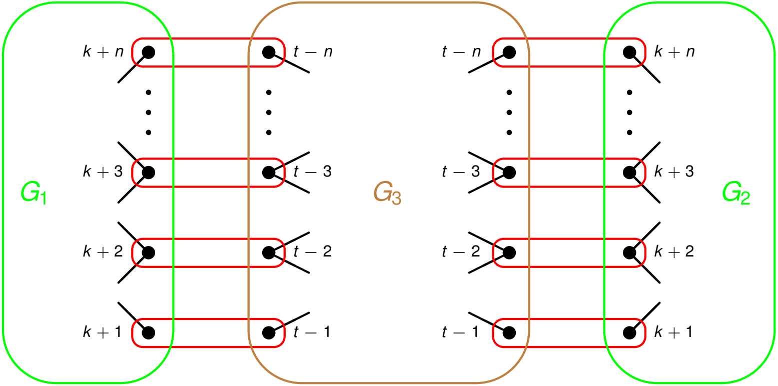
The main idea of the construction



The basic result

Theorem

Let G_1, G_2 be disjoint graphs each of order n and let G_3 be a balanced bipartite graph of order $2n$ with parts U_1 and U_2. Let f be a DC-labelling of $G_1 \cup G_2$ with respect to $(V(G_1), V(G_2))$ and let g be a DC-labelling of G_3 with respect to (U_1, U_2). If f and g are complementary, then there exists a supermagic graph G such that $V(G) = U_1 \cup U_2$, $G(U_1)$ is isomorphic to G_1, $G(U_2)$ is isomorphic to G_2 and $G(U_1, U_2)$ is isomorphic to G_3.

Complements of bipartite graphs

Theorem

- Let G be a d-regular bipartite graph of order $8k$. The complement of G is a supermagic graph if and only if d is odd.
- Let G be a d-regular bipartite graph of order $2n$, where n is odd and d is even. The complement of G is a supermagic graph if and only if $(n, d) \neq (3, 2)$.
- Let G be a d-regular bipartite graph of order $2n$. If $2d < n$ and $5 \leq n \equiv d \equiv 1 \pmod 2$, then the complement of G is a supermagic graph.

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Joins of graphs

Theorem

Let G_1 and G_2 be disjoint d-regular Hamiltonian graphs of order n. If $d \geq 4$ is even and n is odd, then the join $G_1 \oplus G_2$ is a supermagic graph.

Corollary

Let G_1 and G_2 be disjoint d-regular graphs of order n. If $2d \geq n$, $5 \leq n \equiv d \equiv 1 \pmod 2$ and $4 \leq d \equiv 0 \pmod 2$, then the join $G_1 \oplus G_2$ is a supermagic graph.

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Non–regular supermagic graphs

Theorem

Let G_i, $i \in \{1, 2\}$, be a d_i-regular Hamiltonian graph of order n. If $4 \leq d_1 \equiv 0 \pmod 4$, $d_1 = d_2 + 2$ and n is odd, then the join $G_1 \oplus G_2$ is a supermagic graph.

Corollary

Let G_i, $i \in \{1, 2\}$, be a d_i-regular graph of odd order n. If $4 \leq d_1 \equiv 0 \pmod 4$, $d_1 = d_2 + 2$ and $2d_2 \geq n$, then the join $G_1 \oplus G_2$ is a supermagic graph.

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Thank you very much for your attention