Facial non-repetitive colourings of plane graphs

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joint work with Jochen Harant, Frédéric Havet, Roman Soták, Erika Škrabuľáková

Európska únia
Európsky fond regionálneho rozvoja

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Planar graphs

A sequence $r_1, r_2, \ldots, r_{2n}$ such that $r_i = r_{n+i}$ for all $1 \leq i \leq n$, is called a *repetition*. A sequence S is called *non-repetitive* if no subsequence of consecutive terms of S is a repetition.

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Example

$1, 2, 1, 3, 1, 4, 3, 2, 1, 2$ is non-repetitive.

$1, 2, 1, 3, 1, 4, 3, 1, 4, 2$ is repetitive.

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$1, 2, 1, 3, 1, 4, 3, 1, 4, 2$ is repetitive.

Theorem (Thue 1906)

There is an arbitrarily long non-repetitive sequence that is formed using three symbols.

NON-REPETITIVE EDGE-COLOURINGS

Thue chromatic index

Definition (Alon, Grytczuk, Haluszczak, Riordan, 2002)

An edge-colouring of a graph G is a *non-repetitive* if the sequence of colours on any path in G is non-repetitive.

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Theorem (Thue, 1906)

$\pi'(P_n) = 3$ *for every* $n \geq 5$.

Thue chromatic index

Theorem (Currie 2002)

$$\pi'(C_n) = \begin{cases} 4 \text{ \textit{for} } n \in \{5, 7, 9, 10, 14, 17\} \\ 3 \text{ \textit{else.}} \end{cases}$$

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Theorem (Alon, Grytczuk, Haluszczak, Riordan, 2002)

Let G be a simple graph. Then

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Theorem (Alon, Grytczuk, Haluszczak, Riordan, 2002)

Let G be a simple graph. Then

- $\Delta(G) \leq \chi'(G) \leq \pi'(G) \leq c'\Delta(G)^2$ *for some constant* c'.

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- $\Delta(G) \leq \chi'(G) \leq \pi'(G) \leq c'\Delta(G)^2$ *for some constant* c'.
- $\pi'(T) \leq 4(\Delta(T) - 1)$ *if* T *is a tree.*

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- $\pi'(T) \leq 4(\Delta(T) - 1)$ *if* T *is a tree.*
- $\pi'(K_n) \leq 2n$.

Thue chromatic index

Conjecture (Grytczuk, 2008)

There is an absolute constant c such that $\pi'(G) \leq c\Delta(G)$ for any graph G.

FACIAL NON-REPETITIVE EDGE-COLOURINGS

Facial Thue chromatic index

Definition

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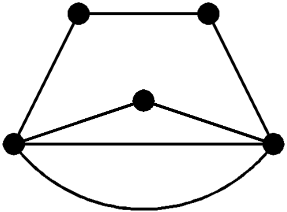
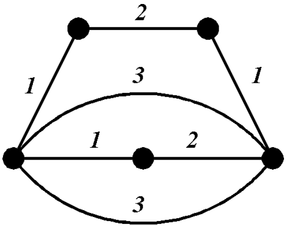
A *facial non-repetitive edge-colouring* of G is an edge-colouring of G such that any facial trail is non-repetitive.

The *facial Thue chromatic index*, denoted $\pi'_f(G)$, is the minimum number of colours of a facial non-repetitive edge-colouring of G.

Facial Thue chromatic index

The facial Thue chromatic index of the graph depends on the embedding of the graph.





Facial Thue chromatic index

Theorem

Let G be a 2-edge-connected plane graph and G^* be the dual of G. Then $\pi'_f(G) \leq \chi'(G^*)$.

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Theorem (Havet, Jendroľ, Soták, Škrabuľáková, 2009)

Let G be a plane triangulation. Then $\pi'_f(G) = 3$.

Trees

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An *almost even tree* is a tree in which every internal vertex, except at most one, is of even degree.

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Let T be an almost even tree. Then $\pi'_f(T) \leq 3$.

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Let T be an almost even tree. Then $\pi'_f(T) \leq 3$.

Theorem (Havet, Jendroľ, Soták, Škrabuľáková, 2009)

Let T be a tree. Then $\pi'_f(T) \leq 4$. Moreover the bound 4 is tight.

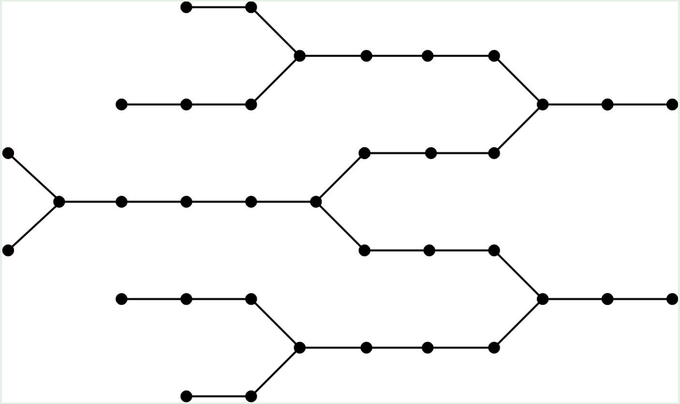
Trees

Theorem (Havet, Jendroľ, Soták, Škrabuľáková, 2009)

Let T be a tree and let K be a subtree of T which is almost even. Then there exists a facial non-repetitive 4-edge-colouring of T that uses only 3 colours on the edges of K.

Trees

Example



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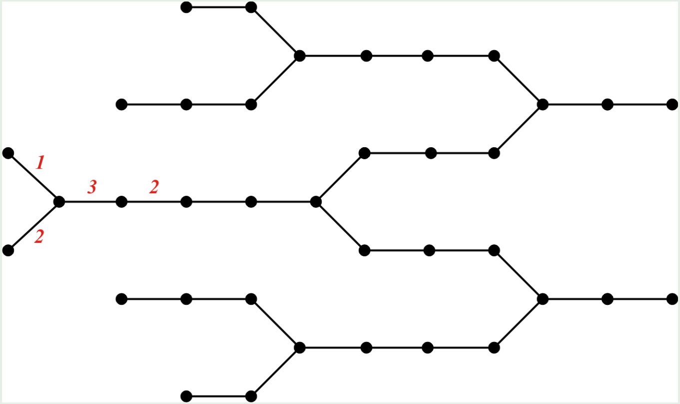

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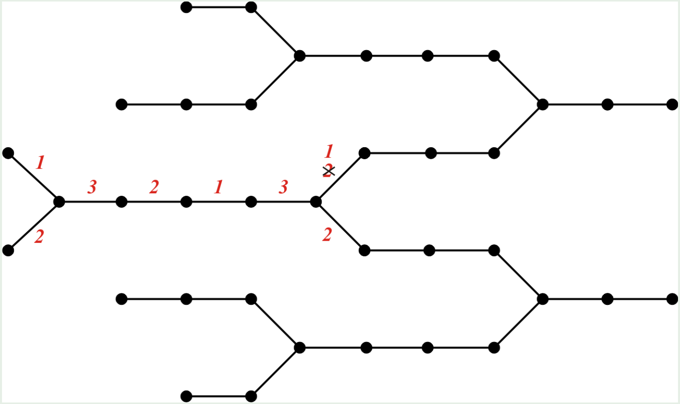

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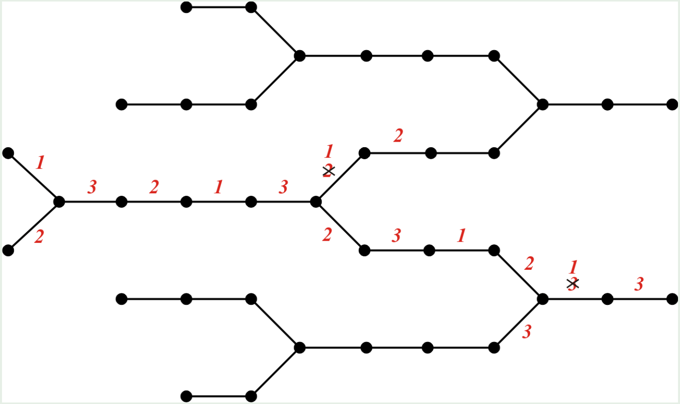

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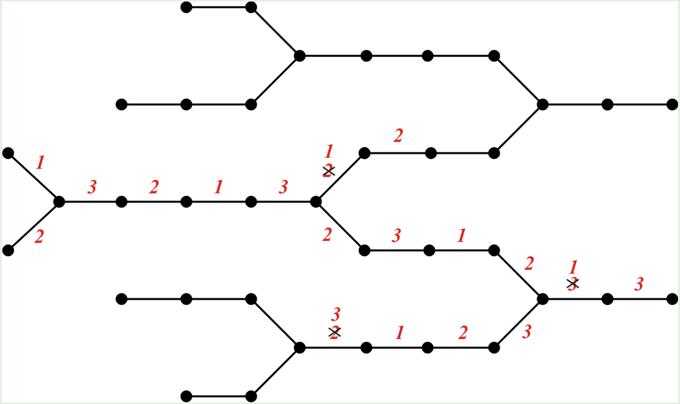

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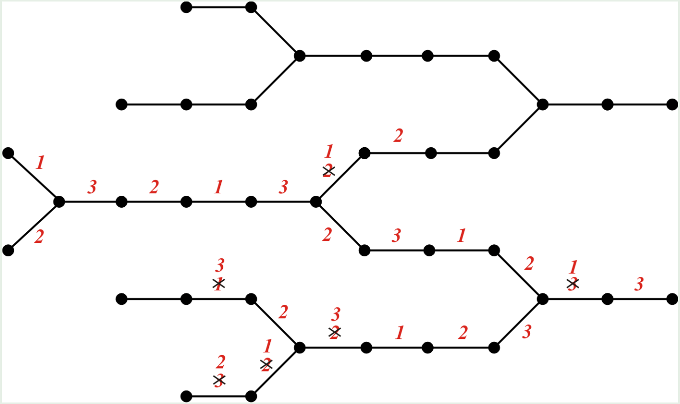

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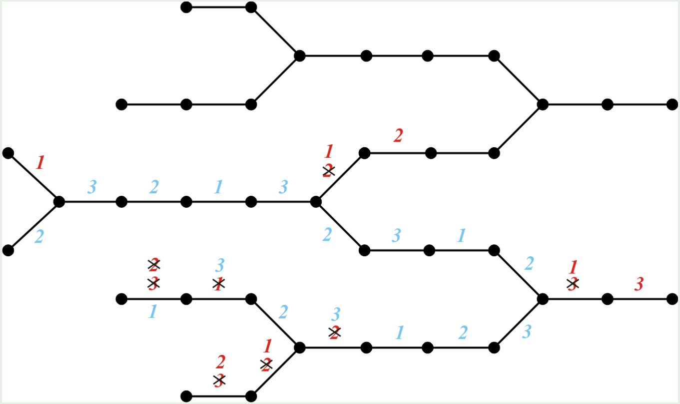

General upper bounds

Theorem (Havet, Jendroľ, Soták, Škrabuľáková, 2009)

Let G be a connected plane graph. Then $\pi'_f(G) \leq 8$.

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Lemma

Let G be a connected plane graph and let T^* be a spanning tree of its dual G^*. Let T be a subgraph of G with edge set $E(T)$ associated to the edge set of T^*. Then $G - E(T)$ is a tree.

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Theorem (Havet, Jendroľ, Soták, Škrabuľáková, 2009)

Let G be a simple 3-connected plane graph. Then $\pi'_f(G) \leq 7$.

General upper bounds

Theorem (Havet, Jendroľ, Soták, Škrabuľáková, 2009)

Let G be a connected plane graph and let its dual contains a Hamilton path. Then $\pi'_f(G) \leq 6$

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Theorem (Havet, Jendroľ, Soták, Škrabuľáková, 2009)

Let G be a Halin graph. Then $\pi'_f(G) \leq 6$

Plane hamiltonian graphs

Let G be a plane hamiltonian graph and H a Hamilton cycle in it. Denote by G_1 (resp.G_2) the subgraph induced by H and the edges of G inside (resp. outside) of H. Evidently G_i, $i = 1, 2$, is a 2-connected outerplanar graph. Let T_i, be the weak dual of G_i, $i = 1, 2$, and $\Delta(T_i)$ its maximum degree.

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Theorem (Havet, Jendroľ, Soták, Škrabuľáková, 2009)

Let G be a hamiltonian plane graph. Then $\pi'_f(G) \leq 7$.

Moreover, if $\max\{\Delta(T_1), \Delta(T_2)\} \leq 4$ then $\pi'_f(G) \leq 6$,

and if $\max\{\Delta(T_1), \Delta(T_2)\} \leq 2$ then $\pi'_f(G) \leq 5$.

Outerplanar graphs

Theorem (Havet, Jendroľ, Soták, Škrabuľáková, 2009)

Let G be a 2-connected outerplanar graph. Then $\pi'_f(G) \leq 7$. Moreover, denoting by T the weak dual of G the following holds:

- *if G is a cycle and $|V(G)| = 2$ then $\pi'_f(G) = 2$;*
- *if G is a cycle and $|V(G)| \notin \{2, 5, 7, 9, 10, 14, 17\}$ then $\pi'_f(G) = 3$;*
- *if G is a cycle and $|V(G)| \in \{5, 7, 9, 10, 14, 17\}$ then $\pi'_f(G) = 4$;*
- *if $\Delta(T) \leq 2$ then $\pi'_f(G) \leq 5$;*
- *if $\Delta(T) \leq 4$ then $\pi'_f(G) \leq 6$.*

Discussion

Problem

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NON-REPETITIVE VERTEX-COLOURINGS

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Theorem (Pezarski, Zmarz, 2009)

Every graph has a subdivision that can be non-repetively coloured using 3 colours.

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Theorem (Grytczuk, 2007)

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Conjecture (Grytczuk, 2007)

There exists an absolute constant k such that any planar graph has a non-repetitive vertex k-colouring.

Theorem (Brešar, Grytczuk, Klavžar, Niwczyk, Peterin, 2007)

Let T be a tree. Then $\pi(T) \leq 4$. Moreover, the bound 4 is tight.

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Theorem (Brešar, Grytczuk, Klavžar, Niwczyk, Peterin, 2007)

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Let G be an outerplanar graph. Then $\pi(G) \leq 12$.

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Theorem (Kundgen, Pelsmajer, 2008)

Let G be a graph of tree-width $t \geq 0$. Then $\pi(G) \leq 4^t$.

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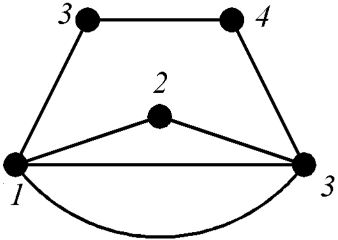
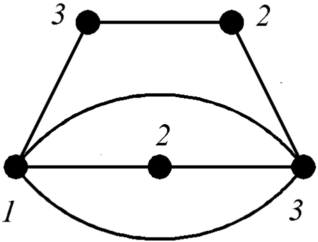
The *facial Thue chromatic number*, denoted $\pi_f(G)$, is the minimum number of colours of a facial non-repetitive vertex-colouring of G.

Facial Thue chromatic number - examples





Facial non-repetitive vertex-colouring of graphs

Conjecture (Harant, Jendrol', 2010)

There exists an absolute constant C such that any plane graph has a facial non-repetitive vertex C-colouring.

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Theorem (Harant, Jendrol', 2010)

Let G be a plane triangulation. Then $3 \leq \pi_f(G) = \chi_0(G) \leq 4$.

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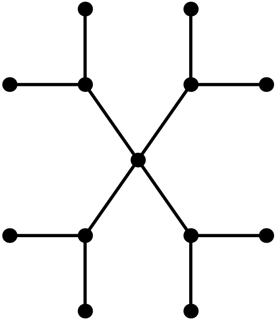

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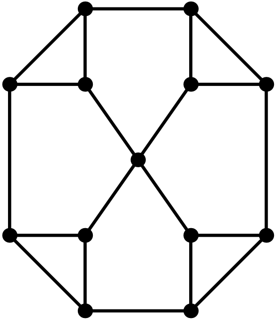

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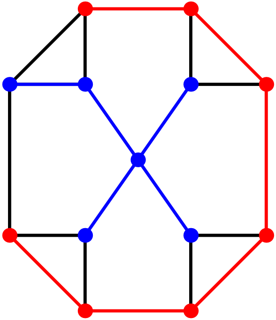

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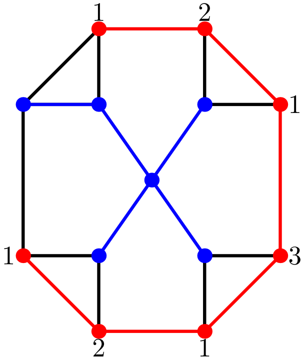

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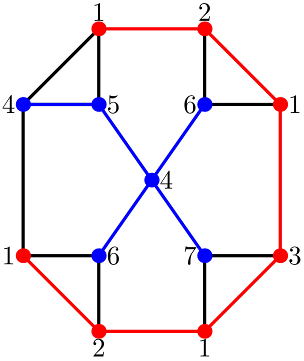

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Theorem (Harant, Jendrol', 2010)

Let G be a hamiltonian plane graph. Then $\pi_f(G) \leq 16$.

Moreover, if no face of G has its size in $\{5, 7, 9, 10, 14, 17\}$, then $\pi_f(G) \leq 9$.

Facial Thue chromatic number of plane hamiltonian graphs

Lemma

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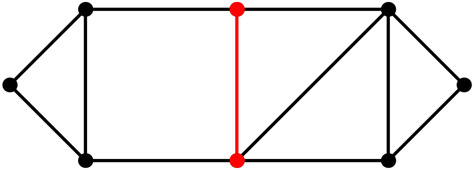

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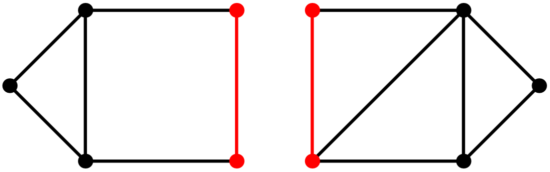

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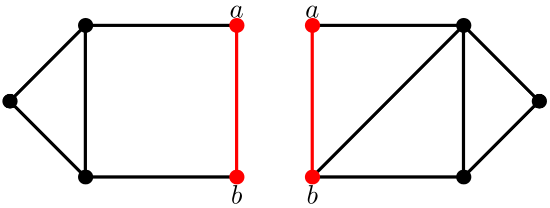
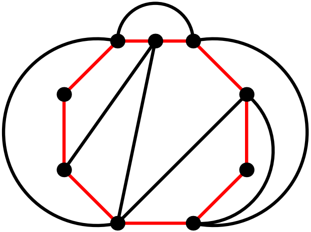
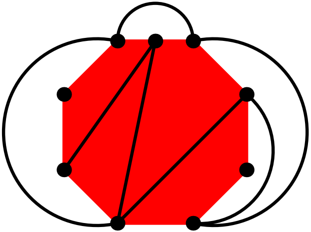
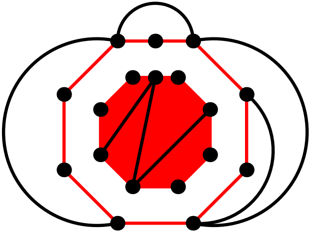
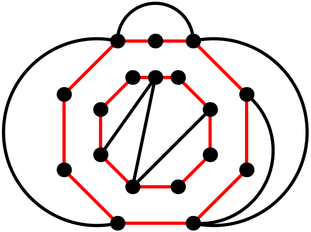


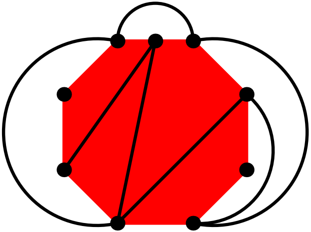


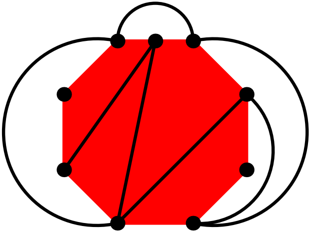




Let $f_i(v)$ be a colour obtained by a vertex v in a $4-$colouring of G_i according the Lemma. Then we colour the vertex v of G with the ordered pair $(f_1(v), f_2(v))$. Clearly this colouring has requred properties.







Theorem (Harant, Jendrol', 2010)

Let G be a 4-connected plane graph. Then $\pi_f(G) \leq 16$.

Moreover, if no face of G has its size in $\{5, 7, 9, 10, 14, 17\}$, then $\pi_f(G) \leq 9$.

Nonrepetitive vertex-colouring of cubic plane graphs

Theorem (Harant, Jendrol', 2010)

Let G be a 2-connected cubic plane graph. Then $\pi_f(G) \leq 112$.

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Theorem (Harant, Jendrol', 2010)

Let G be a 2-connected cubic plane graph all faces of which are multi-4-gonal. Then $\pi_f(G) \leq 27$.

Theorem (Harant, Jendrol', 2010)

Let G be a 2-connected cubic plane graph. Then

$$\pi_f(G) \leq 256.$$

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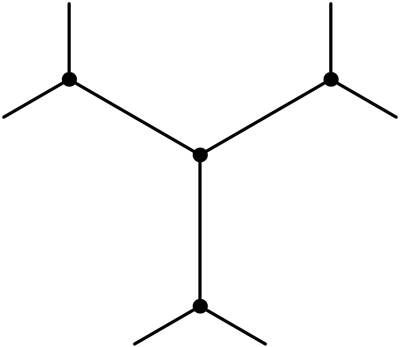

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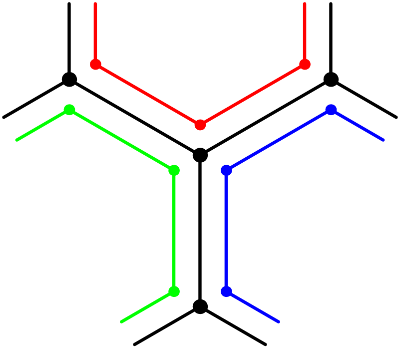
$$\pi_f(G) \leq 256.$$



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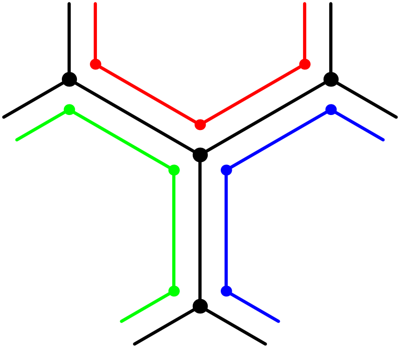

Thanks for your attention