Acyclic edge coloring of planar graphs

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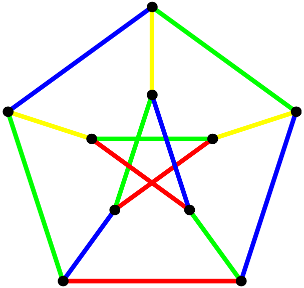

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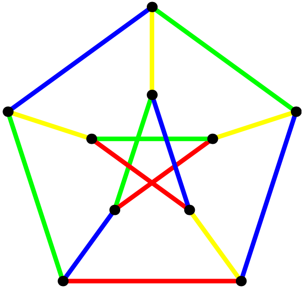

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Theorem (Alon, Sudakov, and Zaks 2001)

$\chi'_a(G) \leq \Delta(G) + 2$ *for all* d*-regular graphs* G *with girth at least* $c\Delta(G) \log \Delta(G)$.

Theorem (Fiedorowicz, Hałuszczak, and Narayanan 2008)

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Conjecture (Cohen, Havet, and Müller 2009)

There exists an integer Δ for which every planar graph G with maximum degree $\Delta(G) \geq \Delta$ admits an acyclic edge coloring with $\Delta(G)$ colors.

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Theorem (Yu, Hou, Liu, Liu, and Xu 2009)

Let G be a planar graph with girth g and maximum degree Δ. Then $\chi'_a(G) = \Delta$ if at least one of the conditions below holds:

1. $\Delta \geq 4$ *and* $g \geq 12$, *or*
2. $\Delta \geq 5$ *and* $g \geq 10$, *or*
3. $\Delta \geq 6$ *and* $g \geq 8$, *or*
4. $\Delta \geq 12$ *and* $g \geq 7$.

		$\Delta(G)$							
		3	4	5	6	...	10	11	12
$g(G)$	3	$\Delta + 12$							
	4	$\Delta + 6$							
	5	$\Delta + 2$							
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Let G be a planar graph with girth g and maximum degree Δ. Then $\chi'_a(G) = \Delta$ if one of the following conditions holds:

1. $\Delta \geq 3$ *and* $g \geq 12$, *or*
2. $\Delta \geq 4$ *and* $g \geq 8$, *or*
3. $\Delta \geq 5$ *and* $g \geq 7$, *or*
4. $\Delta \geq 6$ *and* $g \geq 6$, *or*
5. $\Delta \geq 10$ *and* $g \geq 5$.

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1. Inspect the configurations that are reducible.
2. Consider a minimal counterexample G. Assign the charge to vertices, edges, and faces of G in such a way that its overall sum is negative.
3. Move the charge inside the graph G without changing its sum in such a way that the charge of all elements of G becomes non-negative, unless there is a reducible configuration.

Let $g(G) \geq 8$ and $\Delta(G) = 4$. Then G does not contain a 3-vertex with neighbors of degrees 2, 2, and 3.

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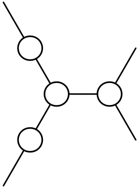

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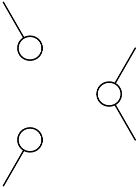

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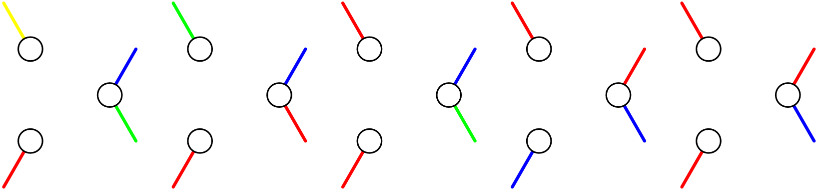

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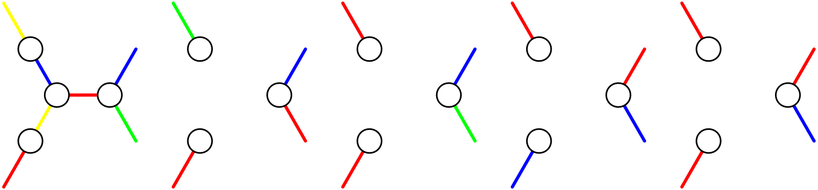

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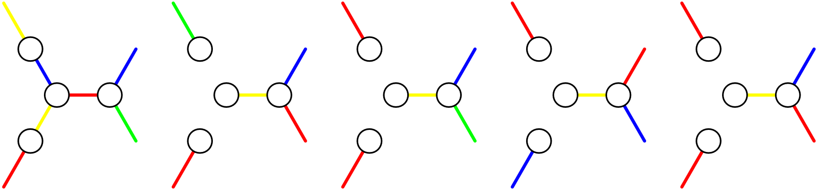

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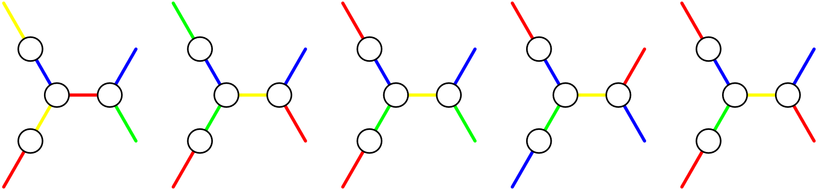

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Let u and v be a pair of subadjacent vertices. If $d(v) < \Delta$, then the number of 2-vertices adjacent to v is at most $d(v) + d(u) - \Delta - 1$.

Lemma

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Let the initial charge be set as follows:

- $w(v) = 5d(v) - 14$ for each vertex v of G;
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By Euler's formula we have that the sum of charges of vertices and faces is -28.
It is clear that since $g \geq 7$ all the faces have nonnegative charge. Vertices of degree 5 have charge 11, vertices of degree 4 have charge 6, vertices of degree 3 have charge 1, and vertices of degree 2 have charge -4.

Let v be a 2-vertex with neighbors v_1 and v_2 such that $d(v_1) \leq d(v_2)$.

(i) If $d(v_1) = 2$, then v sends 0 of charge to v_1 and -4 of charge to v_2.

(ii) If $d(v_1) = 3$, then v sends $-\frac{1}{3}$ of charge to v_1 and $-\frac{11}{3}$ of charge to v_2.

(iii) If $d(v_1) \geq 4$, then v sends -2 of charge both to v_1 and v_2.

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Since $\Delta = 5$, for each 2-vertex with neighbors with degrees d_1 and d_2 we have $d_1 + d_2 \geq \Delta + 2 = 7$.

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Since $\Delta = 5$, for each 2-vertex with neighbors with degrees d_1 and d_2 we have $d_1 + d_2 \geq \Delta + 2 = 7$.
It is easy to see that 2-vertices send all their negative charge to their neighbors of degree at least 3.

Let v be a 3-vertex in G. Its initial charge is 1. By (ii) it receives $-\frac{1}{3}$ of charge from each its 2-neighbor, hence its charge is at least $1 - 3 \cdot \frac{1}{3} = 0$.

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If v is not subadjacent to any ≤ 3-vertex, then it can have at most three 2-neighbors, from which it receives -2 of charge by (iii). Its charge is (at least) $6-3\cdot 2=0$.

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If v is not subadjacent to any 2-vertex and v is subadjacent to a 3-vertex, then it has at most three 2-neighbors, which send at most $-\frac{11}{3}$ of charge each. The charge of v is at least $11 - 3 \cdot \frac{11}{3} = 0$.

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If v is not subadjacent to any 2-vertex and v is subadjacent to a 3-vertex, then it has at most three 2-neighbors, which send at most $-\frac{11}{3}$ of charge each. The charge of v is at least $11 - 3 \cdot \frac{11}{3} = 0$.
If v is not subadjacent to any ≤ 3-vertex, then all its 2-neighbors send -2 of charge by (iii); the charge of v is at least $11 - 5 \cdot 2 = 1 \geq 0$.

Thank you for your attention!