Generalized fractional and circular total coloring of graphs

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Nový Smokovec 2010

Podporujeme výskumné aktivity na Slovensku/
Projekt je spolufinancovaný zo zdrojov EÚ

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Graph properties

- *additive*
- *hereditary*

Definition

Completeness of $\mathcal{P}$:

$$c(\mathcal{P}) = \sup\{k : K_{k+1} \in \mathcal{P}\}$$

Definition

$$\begin{aligned}
\mathcal{O} &= \{G \in \mathcal{I} : G \textit{ is edgeless, i.e. } E(G) = \emptyset\} \\
\mathcal{O}_k &= \{G \in \mathcal{I} : \textit{each component of } G \textit{ has at most } k+1 \textit{ vertices}\} \\
\mathcal{D}_k &= \{G \in \mathcal{I} : \delta(G) \le k \ \textit{ for each } H \subseteq G\} \\
\mathcal{I}_k &= \{G \in \mathcal{I} : G \textit{ contains no } K_{k+2}\}
\end{aligned}$$

Definition

Let $\mathcal{P} \supseteq \mathcal{O}$ and $\mathcal{Q} \supseteq \mathcal{O}_1$ be additive and hereditary graph properties. The $(\mathcal{P}, \mathcal{Q})$-total coloring of a graph G is a coloring of the vertices and edges of G such that, for any color i, it holds $G[V_i] \in \mathcal{P}$, $G[E_i] \in \mathcal{Q}$ and incident vertices and edges are colored differently.

Definition

The $(\mathcal{P}, \mathcal{Q})$-chromatic number $\chi''_{\mathcal{P},\mathcal{Q}}(G)$:

$$\chi''_{\mathcal{P},\mathcal{Q}}(G) = min\{k : G \text{ has a } (\mathcal{P}, \mathcal{Q})\text{-total coloring}\}.$$

Definition

Let $r, s \in \mathbb{N}$. The $(\mathcal{P}, \mathcal{Q})$-total fractional / circular (r, s)-coloring of a simple graph G is a coloring of the vertices and edges of G by arbitrary / consecutive s-element subsets of $\mathbb{Z}_r$ such that, for each color i, the vertices colored by sets containing i induce a subgraph of property $\mathcal{P}$, the edges colored by sets containing i induce a subgraph of property $\mathcal{Q}$, and incident vertices and edges are assigned with disjoint sets.

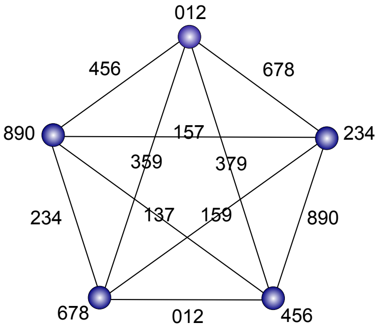
Definition

The fractional / circular $(\mathcal{P}, \mathcal{Q})$-total chromatic number of G:

$$\chi''_{f,\mathcal{P},\mathcal{Q}}(G) = \inf\{\frac{r}{s} : G \text{ has a } (\mathcal{P}, \mathcal{Q})\text{-total fractional } (r, s)\text{-coloring}\}$$

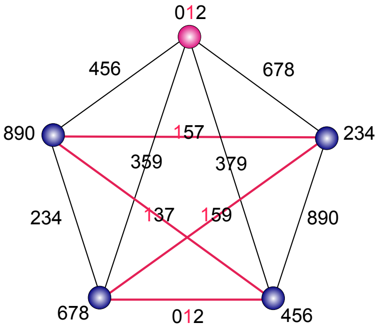
$$\chi''_{c,\mathcal{P},\mathcal{Q}}(G) = \inf\{\frac{r}{s} : G \text{ has a } (\mathcal{P}, \mathcal{Q})\text{-total circular } (r, s)\text{-coloring}\}$$

$$\chi''_{f,\mathcal{P},\mathcal{Q}}(G) \leq \chi''_{c,\mathcal{P},\mathcal{Q}}(G) \leq \chi''_{\mathcal{P},\mathcal{Q}}(G).$$



$$\chi''_{f,\mathcal{O}_1,\mathcal{I}_1}(G) = \frac{10}{3}$$

$$\chi''_{f,\mathcal{P},\mathcal{Q}}(G) \leq \chi''_{c,\mathcal{P},\mathcal{Q}}(G) \leq \chi''_{\mathcal{P},\mathcal{Q}}(G).$$



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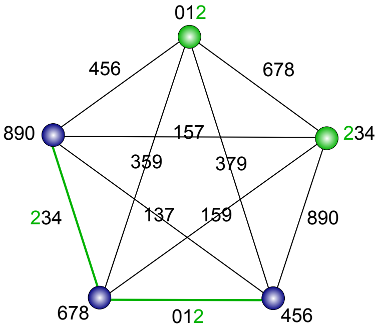

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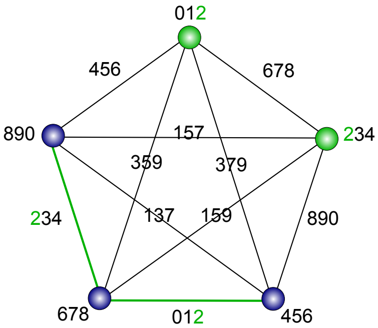
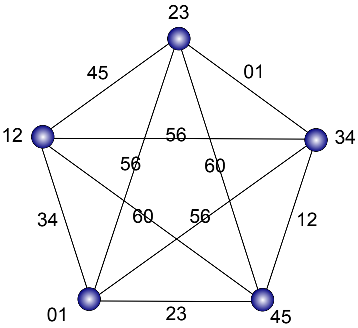
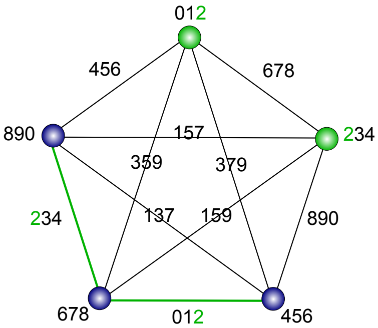
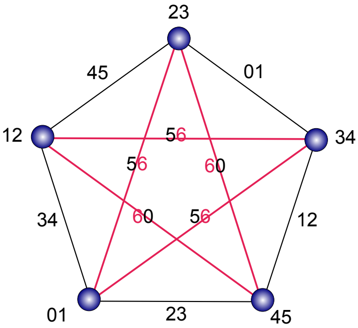




$\chi''_{f,\mathcal{O}_1,\mathcal{I}_1}(G) = \frac{10}{3}$ $\chi''_{c,\mathcal{O}_1,\mathcal{I}_1}(G) = \frac{7}{2}$

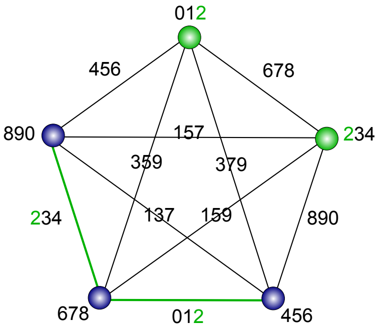
$$\chi''_{f,\mathcal{P},\mathcal{Q}}(G) \leq \chi''_{c,\mathcal{P},\mathcal{Q}}(G) \leq \chi''_{\mathcal{P},\mathcal{Q}}(G).$$

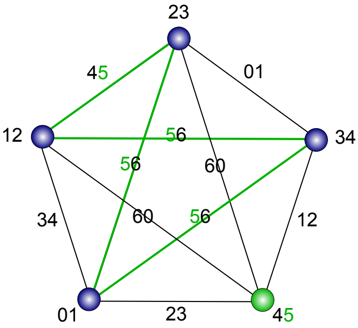




$$\chi''_{f,\mathcal{O}_1,\mathcal{I}_1}(G) = \frac{10}{3} \qquad \chi''_{c,\mathcal{O}_1,\mathcal{I}_1}(G) = \frac{7}{2}$$

$$\chi''_{f,\mathcal{P},\mathcal{Q}}(G) \leq \chi''_{c,\mathcal{P},\mathcal{Q}}(G) \leq \chi''_{\mathcal{P},\mathcal{Q}}(G).$$





$\chi''_{f,\mathcal{O}_1,\mathcal{I}_1}(G) = \frac{10}{3}$

$\chi''_{c,\mathcal{O}_1,\mathcal{I}_1}(G) = \frac{7}{2}$

Theorem

$$\chi''_{\mathcal{P},\mathcal{Q}}(G) - 1 < \chi''_{c,\mathcal{P},\mathcal{Q}}(G) \leq \chi''_{\mathcal{P},\mathcal{Q}}(G).$$

Theorem

$$\chi''_{c,\mathcal{P},\mathcal{Q}}(G) = \min\{\tfrac{r}{s} : G \text{ has circular } (\mathcal{P},\mathcal{Q})\text{-total } (r,s)\text{-coloring with } r \leq |V(G)| + |E(G)|\}.$$

Lemma

If $H \subseteq G$ then

$$\chi''_{c,\mathcal{P},\mathcal{Q}}(H) \leq \chi''_{c,\mathcal{P},\mathcal{Q}}(G),$$

$$\chi''_{f,\mathcal{P},\mathcal{Q}}(H) \leq \chi''_{f,\mathcal{P},\mathcal{Q}}(G).$$

Lemma

If $\mathcal{P}_1 \subseteq \mathcal{P}_2$ and $\mathcal{Q}_1 \subseteq \mathcal{Q}_2$ then

$$\chi''_{c,\mathcal{P}_1,\mathcal{Q}_1}(G) \geq \chi''_{c,\mathcal{P}_2,\mathcal{Q}_2}(G),$$

$$\chi''_{f,\mathcal{P}_1,\mathcal{Q}_1}(G) \geq \chi''_{f,\mathcal{P}_2,\mathcal{Q}_2}(G).$$

Theorem

Let $n \geq 3$. *Then* $\chi''_{f,\mathcal{D}_1,\mathcal{D}_1}(K_n) = \chi''_{c,\mathcal{D}_1,\mathcal{D}_1}(K_n) = \frac{n(n+1)}{2(n-1)}$.

Sketch of proof.

We consider a $(\mathcal{D}_1, \mathcal{D}_1)$-total fractional (r, s)-coloring of K_n. This coloring yields $(n-1)r \geq (n + \binom{n}{2})s$ and consequently

$$\chi''_{f,\mathcal{D}_1,\mathcal{D}_1}(K_n) \geq \frac{n(n+1)}{2(n-1)}.$$

Conversely, we will construct (nontrivial) $(\mathcal{D}_1, \mathcal{D}_1)$-total circular $(n(n+1), 2(n-1))$-coloring. □

Generalized edge coloring

Definition

Let $\mathcal{Q} \supseteq \mathcal{O}_1$ be an additive and hereditary graph property. The $(\mathcal{Q}, k)$-edge coloring of a graph G is a k-coloring of the edges of G such that, for any color i, it holds $G[E_i] \in \mathcal{Q}$.

Definition

The $\mathcal{Q}$-chromatic index $\chi'_{\mathcal{Q}}(G)$:

$$\chi'_{\mathcal{Q}}(G) = min\{k : G \text{ has a } (\mathcal{Q}, k)\text{-edge coloring}\}.$$

Theorem

$$\chi'_{\mathcal{I}_k}(K_{(k+1)n}) \leq \chi'_{\mathcal{I}_k}(K_n) + 1.$$

$$\chi'_{\mathcal{I}_k}(K_n) \leq \lceil \log_{k+1}(n) \rceil.$$

(Several initial values for $\mathcal{I}_1$)

$$\chi'_{\mathcal{I}_1}(K_5) = 2$$

$$\chi'_{\mathcal{I}_1}(K_6) = 3$$

$$\chi'_{\mathcal{I}_1}(K_{12}) = 3$$

$$\chi'_{\mathcal{I}_1}(K_{17}) = 4$$

$$\chi'_{\mathcal{I}_1}(K_{24}) = 4$$

Theorem

For each $k, l \in \mathbb{N}$, there exists $T(k)$ such that, for each $n \geq T(k)$ and for each $\mathcal{P}$ with $c(\mathcal{P}) = k$,

$$\chi''_{f,\mathcal{P},\mathcal{I}_l}(K_n) = \chi''_{c,\mathcal{P},\mathcal{I}_l}(K_n) = \frac{n}{k+1}.$$

Sketch of proof.

Let $c(\mathcal{P}) = k$. We consider a $(\mathcal{P}, \mathcal{I}_l)$-total fractional (r, s)-coloring of K_n. Then $r(k+1) \geq ns$ and consequently

$$\chi''_{f,\mathcal{P},\mathcal{I}_l}(K_n) \geq \frac{n}{k+1}.$$

Conversely, we will construct a $(\mathcal{P}, \mathcal{I}_1)$-total circular $(n, k+1)$-coloring. □

Thank you for your attention.